Trigonometry Workbook Solutions

Graphing the trig functions

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MATH

PERIOD AND AMPLITUDE

■ 1. Find all possible sine and cosine functions $a\sin(bx)$ and $a\cos(bx)$ which have the period of 135° and an amplitude of 10.

Solution:

For the sine and cosine functions $a\sin(bx)$ and $a\cos(bx)$, the amplitude and period are

$$\text{amplitude} = |a|$$

$$\text{period} = \frac{360^\circ}{b} = \frac{2\pi}{b}$$

We're told the amplitude is $|a| = 10$, so a could be -10 or 10. We also know that both functions have a period of 135°, so

$$\text{period} = \frac{360^\circ}{b} = 135^\circ$$

$$b = \frac{360^\circ}{135^\circ} = \frac{8}{3}$$

Putting these values together, there are four possible functions that satisfy the given conditions:

$$-10\sin\frac{8x}{3} \qquad\qquad 10\sin\frac{8x}{3}$$

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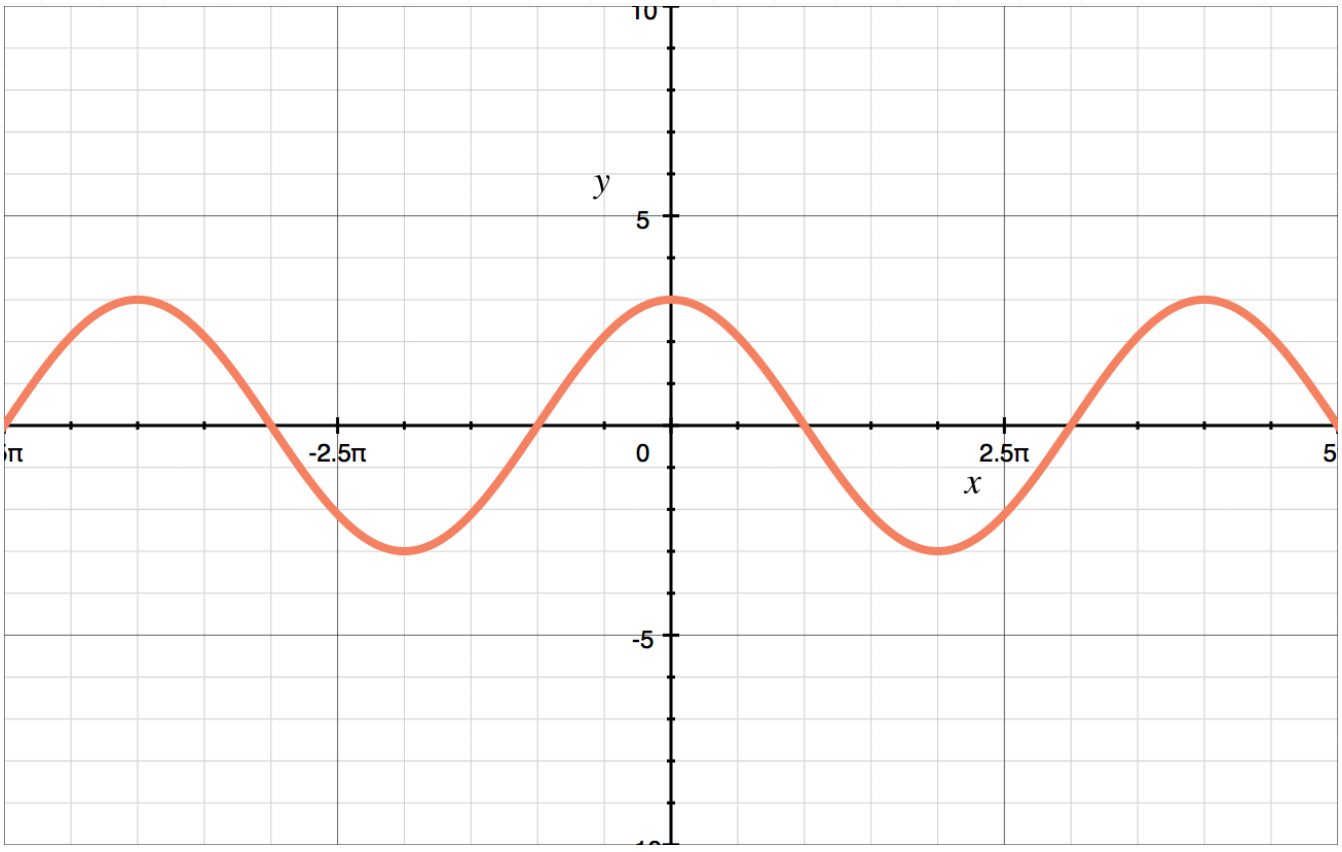
$$-10\cos\frac{8x}{3}$$

$$10\cos\frac{8x}{3}$$

2. Find the equation of the cosine function.



Solution:

We need to build the cosine function $y = a\cos(bx)$ which has

$$\text{amplitude} = |a|$$

$$\text{period} = \frac{2\pi}{b}$$

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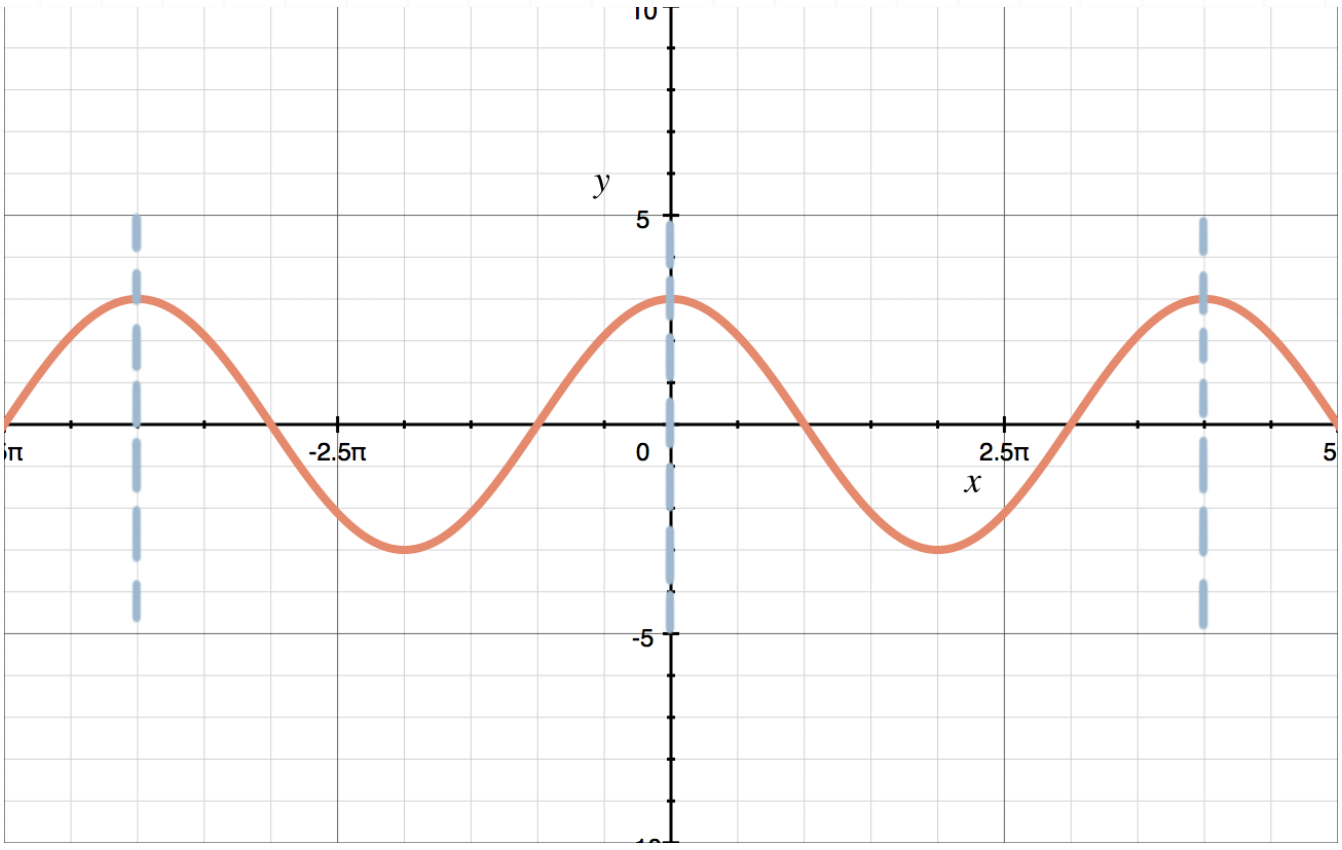
Since the graph shown has range $[-3,3]$, the amplitude is equal to 3, which means $a = -3$ or $a = 3$. Since the function at $x = 0$ is positive, a is also positive, so $a = 3$.

We can see from the graph that the interval over $[0,4\pi]$ repeats, so the period is 4π.



So the equation for b is

$$\text{period} = \frac{2\pi}{b} = 4\pi$$

$$b = \frac{2\pi}{4\pi} = \frac{1}{2}$$

With $a = 3$ and $b = 1/2$, the equation of the cosine function is

$$y = a\cos(bx)$$

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$$y = 3\cos\frac{x}{2}$$

■ 3. Chuck rides a Ferris wheel at a carnival. The wheel has a 24-meter diameter, and turns at one revolution every 3 minutes. Chuck knows that the altitude of his cabin relative to the center of the wheel is modeled by

$$h(t) = a\cos(bt)$$

where $t \geq 0$ is the time in minutes, and $t = 0$ is the time when Chuck started to ride. Find a and b to complete the equation for $h(t)$.

Solution:

The radius of the wheel is half the diameter, so $24/2 = 12$. Which means the altitude of Chuck's cabin relative to the center of the wheel changes from -12 to 12. So the amplitude is 12, and the parameter a can accept the values -12 or 12.

Plug $t = 0$ into $h(t)$.

$$h(0) = a\cos(b \cdot 0) = a$$

We know that the altitude $h(0)$ is the initial position of the Chuck's cabin, which is -12, so $a = -12$.

To find b, let's use the formula for period.

$$\text{period} = \frac{2\pi}{b}$$

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Since the full period (the same as the full revolution of the wheel) is 3 minutes,

$$\frac{2\pi}{b} = 3$$

$$b = \frac{2\pi}{3}$$

So the equation for $h(t)$ is

$$h(t) = -12\cos\frac{2\pi t}{3}$$

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SKETCHING SINE AND COSINE

■ 1. Using set notation, find all values where $f(x)$ is positive.

$$f(x) = -3\sin 2x$$

Solution:

Within a period, the graph of the sine function is positive in a one half of the period, and negative in another. So, (1) let's find the period P of the given function, then (2) find an interval within the first period $[0,P]$ where the function is positive, and finally (3) extend the solution to other periods.

For the sine function $y = a\sin(bx)$, the period P is defined by the argument parameter b as

$$P = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$$

Since the period of the function is π, and since the sign of a is negative, the function is positive on the second half of the period, $(\pi/2,\pi)$. Extend that solution to all periods by adding πn. The function is positive on

$$\left(\frac{\pi}{2} + \pi n,\ \pi + \pi n\right) \text{ where } n \text{ is any integer}$$

$$\left\{\left(\frac{\pi}{2} + \pi n, \pi + \pi n\right) :\ n \in \mathbb{Z}\right\}$$

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■ 2. Lisa rested on the seafront. She noted that the waves have a height of 60 cm from trough to crest (from the lowest to highest point), and that they occur every 8 s. Write an equation that models the height of a particle in water in the form $h(t) = a\sin(bt)$ where t is the time in seconds, and $h(t)$ is the height above the sea level in meters. Assume that $t = 0$ is the moment when a particle is in the middle of the front side of the wave. Find a and b and sketch the graph of the function.

Solution:

Since the height of the waves is 60 cm $= 0.6$ m, the height of a particle relative to sea level changes from -0.3 to 0.3. So the amplitude is 0.3, and the parameter a can accept the values -0.3 or 0.3.

Since at $t = 0$ the value of the function is 0 and the function is increasing (since the particle is in the middle of the front side of the wave, it's going up), the amplitude is $a = 0.3$.

Since we're given the period of the function, 8 s, we can find the value of the parameter b.

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$$\text{period} = \frac{2\pi}{b}$$

$$\frac{2\pi}{b} = 8$$

$$2\pi = 8b$$

$$b = \frac{2\pi}{8} = \frac{\pi}{4}$$

The equation of the function is

$$h(t) = 0.3 \sin\left(\frac{\pi t}{4}\right)$$

To sketch the graph, first examine the behavior of $\sin x$ within its first period, $[0,2\pi]$. The initial position is zero, then it increases and reaches its maximum value at $x = \pi/2$, then decreases to 0 at $x = \pi$, then decreases until it reaches its minimum value at $x = 3\pi/2$, and finally increases to 0 at $x = 2\pi$.

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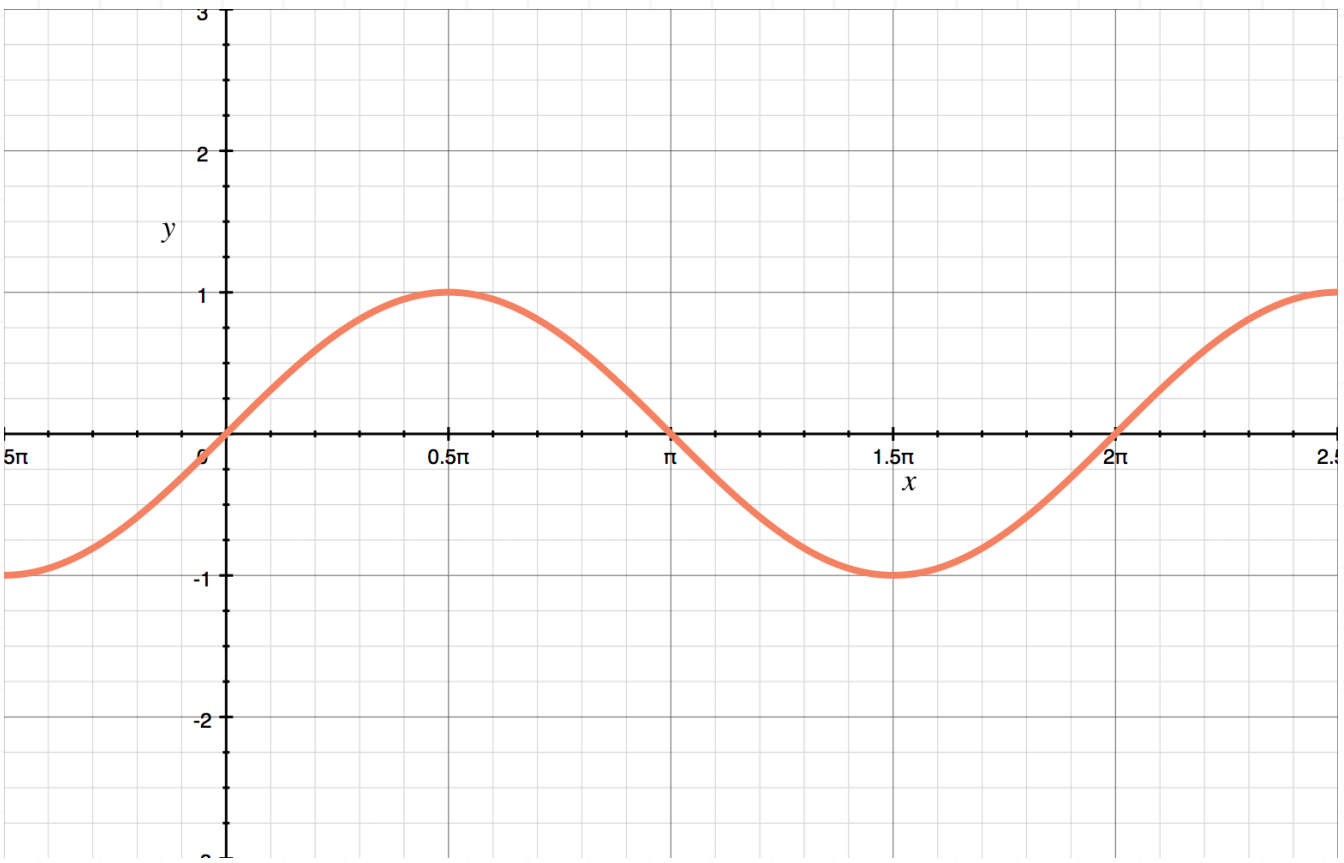

The function $h(t)$ behaves in the same way, but its period is 8 instead of 2π. The initial position is zero, then it increases and reaches its maximum value of 0.3 at $x = 2$, then decreases to zero at $x = 4$, then decreases until it reaches the minimum value at $x = 6$, and finally increases to zero at $x = 8$.

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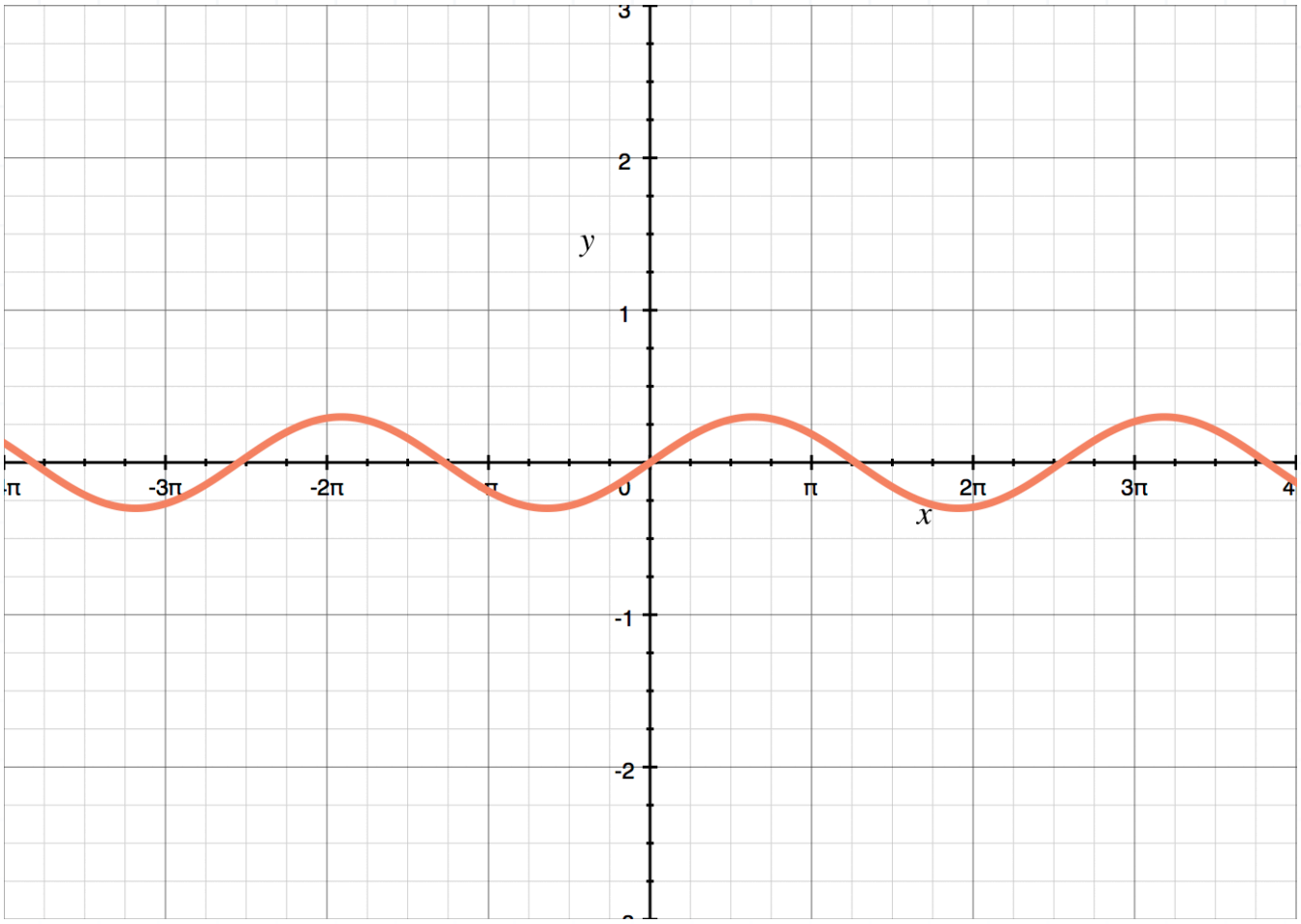
Therefore, we get the graph on the first period, $[0,8]$, and can extend it to other periods. With $a = 0.3$ and $b = \pi/4$, the graph is



■ 3. Donald rides a ferris wheel at a carnival. The altitude of his cabin relative to the center of the wheel is modeled by $h(t) = a\cos(bt)$, with $t \geq 0$, where t is time in minutes. Assume that $t = 0$ is the moment when Donald entered his cabin at the bottom of the wheel. The wheel has a 10-meter radius, and turns at one revolution every two minutes. Find a and b, and sketch the graph of the function.

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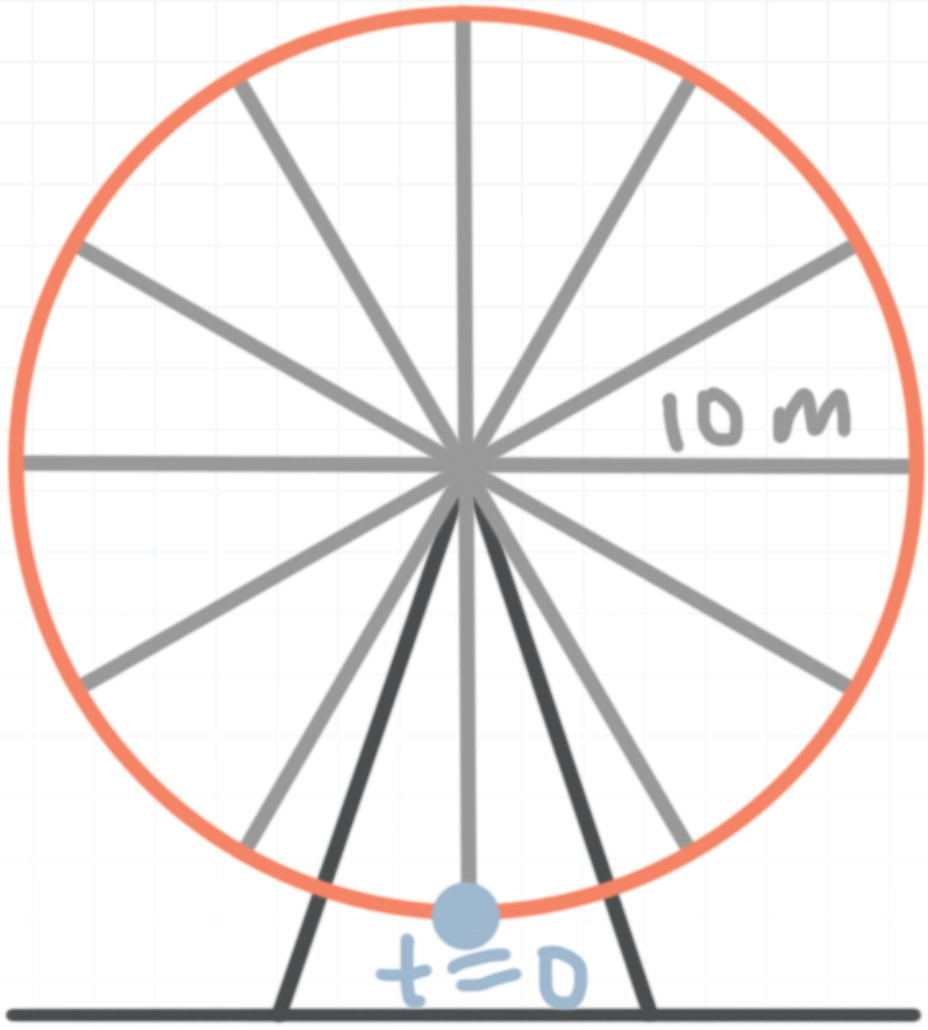

Solution:

The altitude of Donald's cabin relative to the center of the wheel changes from -10 at the bottom to 10 at the top. The amplitude is 10, and the parameter a can accept the values -10 or 10. Since at $t = 0$ the value of the function is negative and equal to -10, the parameter a is also equal to -10.

The period of the function is 2 minutes, so we can find the value of the parameter b.

$$\text{period} = \frac{2\pi}{b}$$

$$2 = \frac{2\pi}{b}$$

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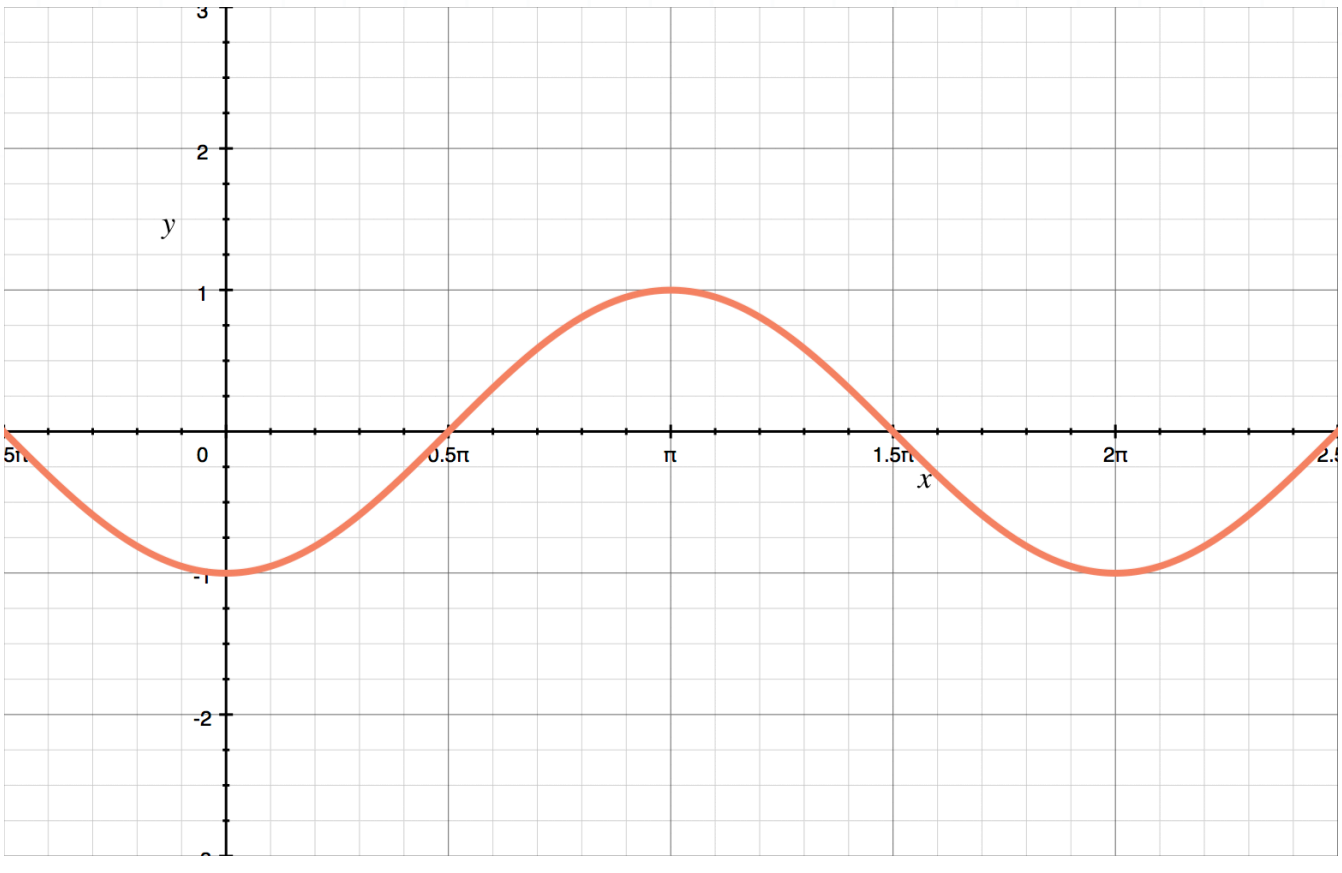
$$2b = 2\pi$$

$$b = \pi$$

The equation of the function is

$$h(t) = -10\cos(\pi t) \text{ with } t \geq 0$$

To sketch the graph, look at the end behavior of $-\cos x$ within its first period, $[0,2\pi]$. It reaches its minimum value at $x = 0$, then increases to zero at $x = \pi/2$, then proceeds to increase until it reaches its maximum value at $x = \pi$, then decreases to zero at $x = 3\pi/2$, and finally decreases until it reaches its minimum value at $x = 2\pi$.



The function $-10\cos(\pi t)$ behaves in the same way, but instead of 2π, it has the period of 2. It reaches its minimum value of -10 at $t = 0$, then increases to zero at $t = 1/2$, then proceeds to increase until it reaches its maximum value of 10 at $t = 1$, then decreases to zero at $t = 3/2$, and finally decreases

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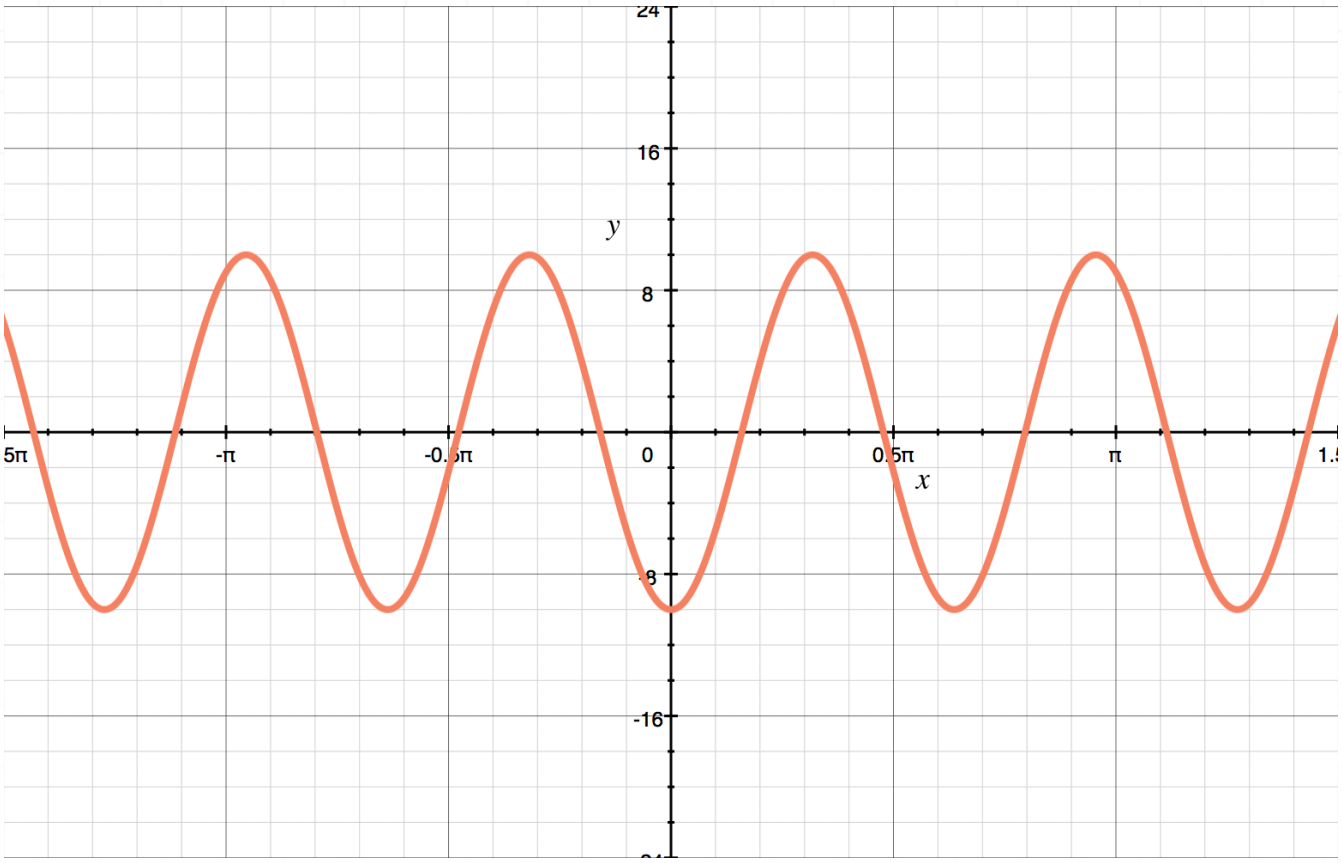
until it reaches the minimum value at $t = 2$. We get the graph on the first period $[0,2]$, we know $a = -10$ and $b = \pi$, and can extend it through all other periods by duplicating.



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HORIZONTAL AND VERTICAL SHIFTS

1. The solutions of the trigonometric equation $f(x) = 0$ are $x \in \{\pi + 3\pi k : \ k \in \mathbb{Z}\}$. Find the set of solutions for each equation.

(a) $f\left(x + \frac{\pi}{2}\right) = 0$

(b) $f(x - 5\pi) = 0$

Solution:

For equation (a), if the value of $\pi/2$ is added to the argument, then the graph of the function is shifted to the left from $f(x)$.

So all zeros of the function are also shifted to the left (in the negative direction of the x-axis) by $\pi/2$. So in order to write down the zeros of the equation, we just need to decrease the zeros of the equation $f(x) = 0$ by $\pi/2$. So the solution set is

$$\left\{\pi - \frac{\pi}{2} + 3\pi k : \ k \in \mathbb{Z}\right\}$$

$$\left\{\frac{\pi}{2} + 3\pi k : \ k \in \mathbb{Z}\right\}$$

For equation (b), if the value of 5π is subtracted from the argument, then the graph of the function is shifted to the right from $f(x)$.

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So all zeros of the function are also shifted to the right (in the positive direction of x-axis) by 5π. So in order to write down zeros of the equation, we just need to increase the zeros of the equation $f(x) = 0$ by 5π. So the solution set is

$$\{\pi + 5\pi + 3\pi n : \ n \in \mathbb{Z}\}$$

$$\{6\pi + 3\pi n : \ n \in \mathbb{Z}\}$$

This solution is correct, but it isn't in its simplest form. In set notation, we usually write the nonnegative solution within the first period, and then the multiple of the period. 6π is not the first nonnegative zero. For $n = -1$, we have 3π. For $n = -2$, we have 0. Let's use that in the final answer.

$$\{0 + 3\pi n : \ n \in \mathbb{Z}\}$$

$$\{3\pi n : \ n \in \mathbb{Z}\}$$

■ 2. Find all values where $y = \sin(x) + 1$ is positive.

Solution:

If the value of 1 is added to $\sin x$, then the graph of the sine function gets shifted up from $\sin x$ by 1. The range of the function $\sin x$ is $[-1,1]$, and since the range is also shifted up by 1, it becomes $[0,2]$. The function is always positive except at $\sin x + 1 = 0$.

$$\sin x + 1 = 0$$

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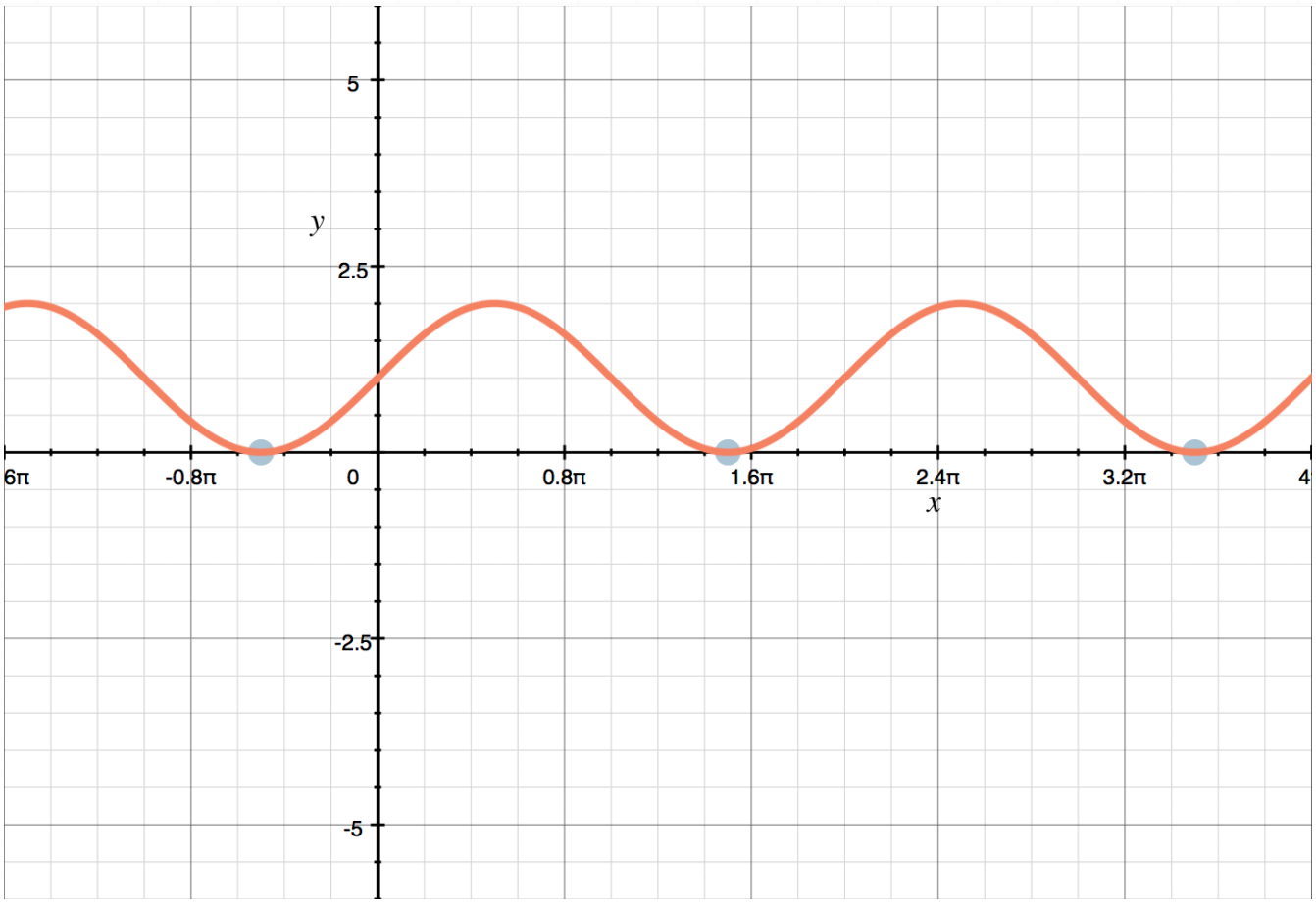
$$\sin x = -1$$

From the unit circle, we know this is only true when $x = 3\pi/2$. We can get the other solutions by adding multiples of 2π. So the zeros of the function are

$$\left\{ \frac{3\pi}{2} + 2\pi n : \ n \in \mathbb{Z} \right\}$$

And a sketch of them looks like this:



We need to write down all of the points except these zeros. The first interval between zeros at $n = 0$ and $n = 1$ is

$$\left(\frac{3\pi}{2}, \frac{3\pi}{2} + 2\pi \right) = \left(\frac{3\pi}{2}, \frac{7\pi}{2} \right)$$

Extend the interval by adding multiples of 2π. The solution set is

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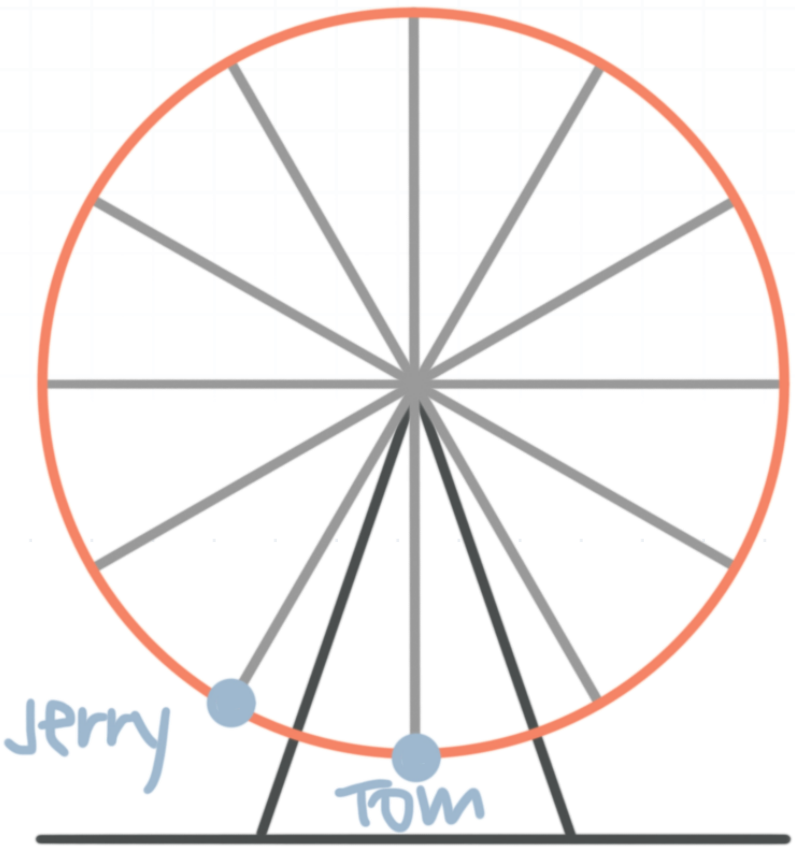
$\left(\frac{3\pi}{2} + 2\pi n, \frac{7\pi}{2} + 2\pi n\right)$ where n is any integer

■ 3. Tom and Jerry ride a 12-cabin Ferris wheel at a carnival. Jerry sat in the first cabin, but Tom was late and sat in the second. The altitude of Jerry's cabin with respect to the center of the wheel is modeled by $h(t) = -6\cos(t)$ with $t \geq 0$, where t is the time in minutes, and $t = 0$ is the moment when Jerry gets in his cabin. Find the equation that models the altitude of Tom's cabin and sketch the graph of the function.



Solution:

Tom's cabin repeats all of the moves of Jerry's cabin, but with a delay. And in terms of the graphs of their equations, this delay means nothing more

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than a horizontal shift by some angle. This angle can be calculated as the angle between the terminal sides of Jerry's and Tom's cabins.

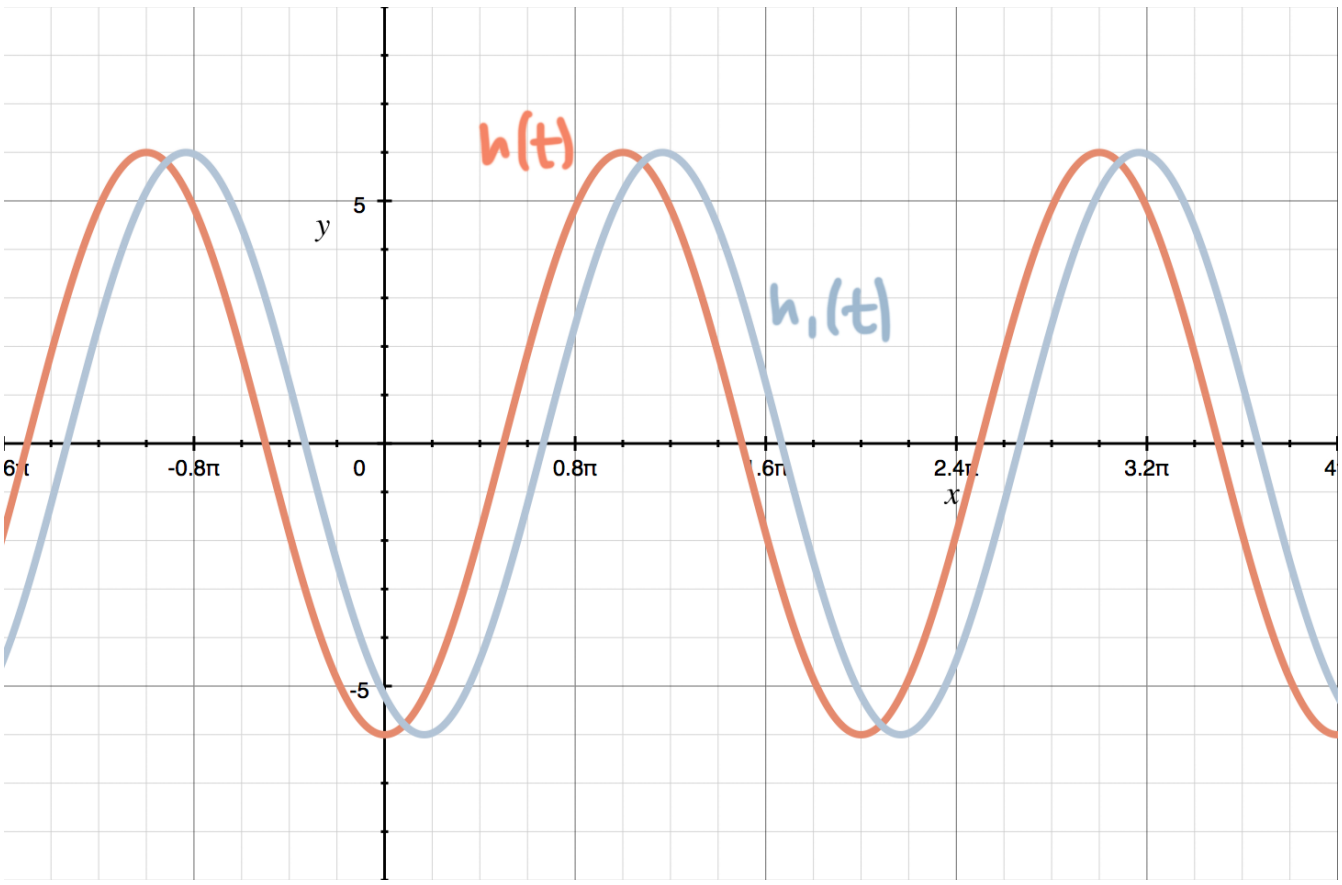
Since the wheel has 12 cabins, all at equal distances from each other, and since their corresponding 12 angles form a complete angle of 2π, each angle is equal to

$$\frac{2\pi}{12} = \frac{\pi}{6}$$

So while Jerry's movement is described by $h(t) = -6\cos t$, Tom's movement with a delay of $\pi/6$ is modeled by

$$h_1(t) = -6\cos\left(t - \frac{\pi}{6}\right) \text{ with } t \geq \frac{\pi}{6}$$

To sketch the graph of $h_1(t)$, sketch $h(t)$ with an amplitude of 6, and then shift it to the right by $\pi/6$.



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SEQUENCE OF TRANSFORMATIONS

1. What are the period, amplitude, domain, and range of the function?

$$y = -3\cos(2\pi x - 1) + 4$$

Solution:

Remember that a vertical transformation doesn't affect a function's period, and a horizontal transformation doesn't affect a function's amplitude or range.

Start by finding the period. Plugging $b = 2\pi$ from the function, the period of the cosine function is

$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{2\pi} = 1$$

Find the domain. For the cosine function, the domain is the set of all real numbers, or

$$\text{domain} = (-\infty, \infty)$$

Find the amplitude and range. The amplitude of $\cos\theta$ is 1, and the range is $[-1,1]$. After multiplying the function by -3 (the same as 3 in this case), we have an amplitude of 3, and a range of $[-3,3]$. After adding 4, we have the same amplitude of 3, but the range is shifted up by 4.

$$\text{range} = [-3+4, 3+4] = [1,7]$$

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So for the function,

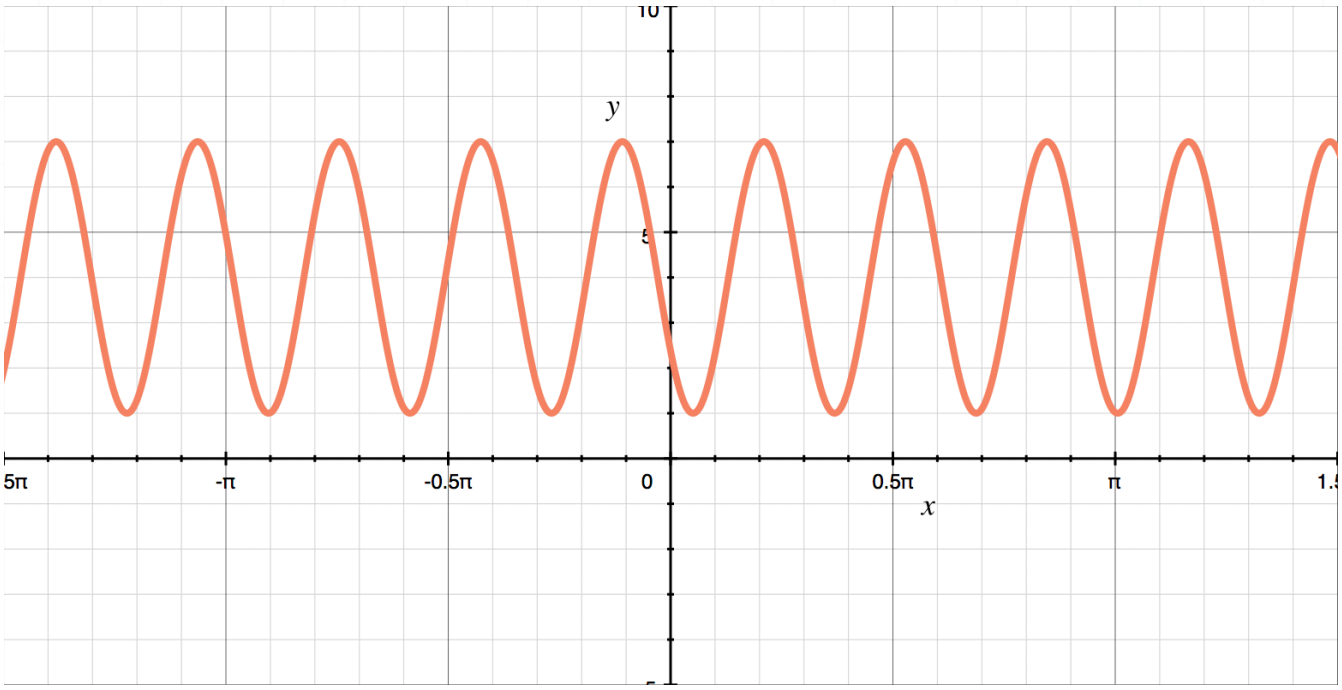
period $= 1$

domain $= (-\infty, \infty)$

amplitude $= 3$

range $= [1,7]$

And a sketch of the graph is



■ 2. Find the equation of $f(x) = \sin(x - \pi)$ after the sequence of transformations:

A horizontal compression by 2, then

A horizontal shift to the right by 3π, then

A vertical stretching by 5, then

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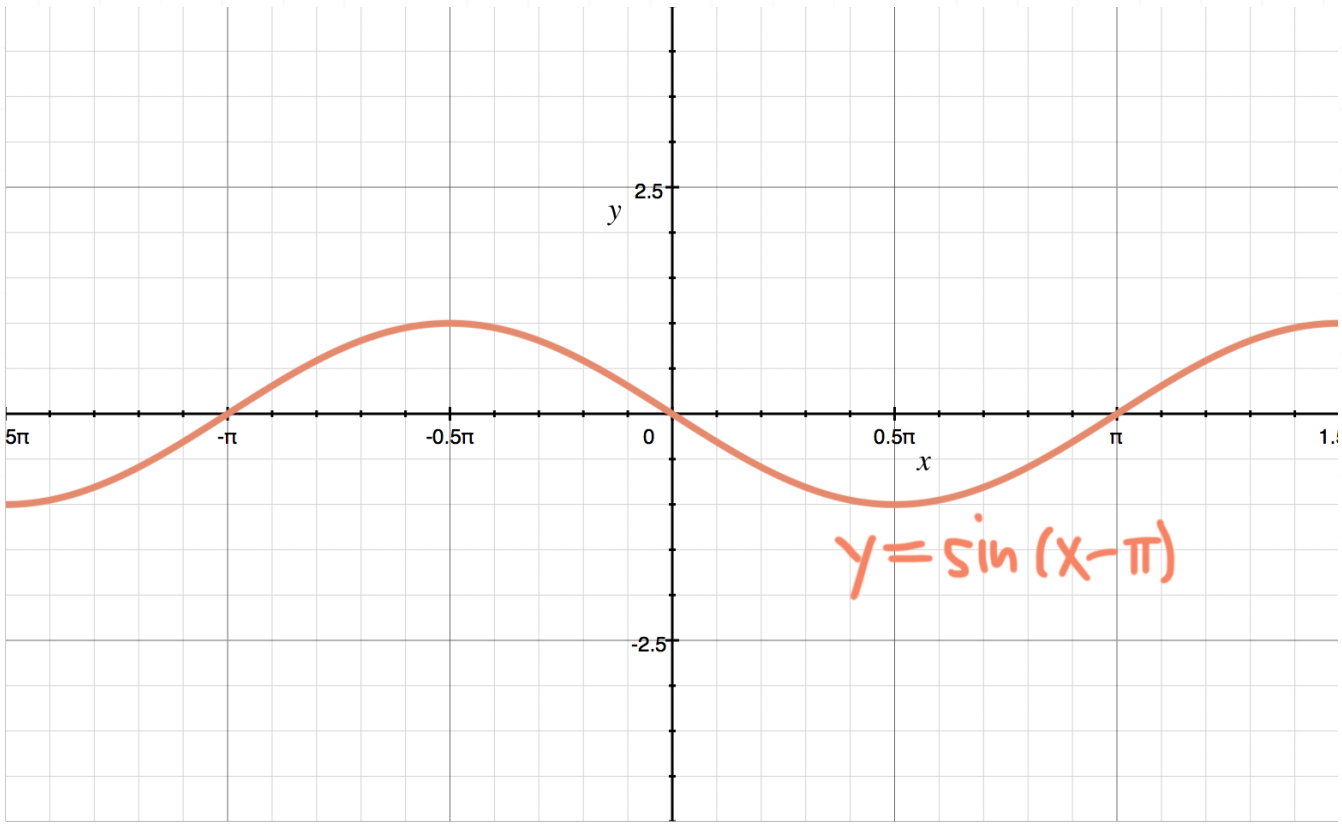
A vertical shift down by 2, then

A reflection about the x-axis.

Solution:

Start with $f(x) = \sin(x - \pi)$ and apply each transformation one-by-one.



Apply a horizontal compression of 2 by multiplying the argument by 2.

$$f_1(x) = f(2x)$$

$$f_1(x) = \sin(2x - \pi)$$

A sketch of the transformation is

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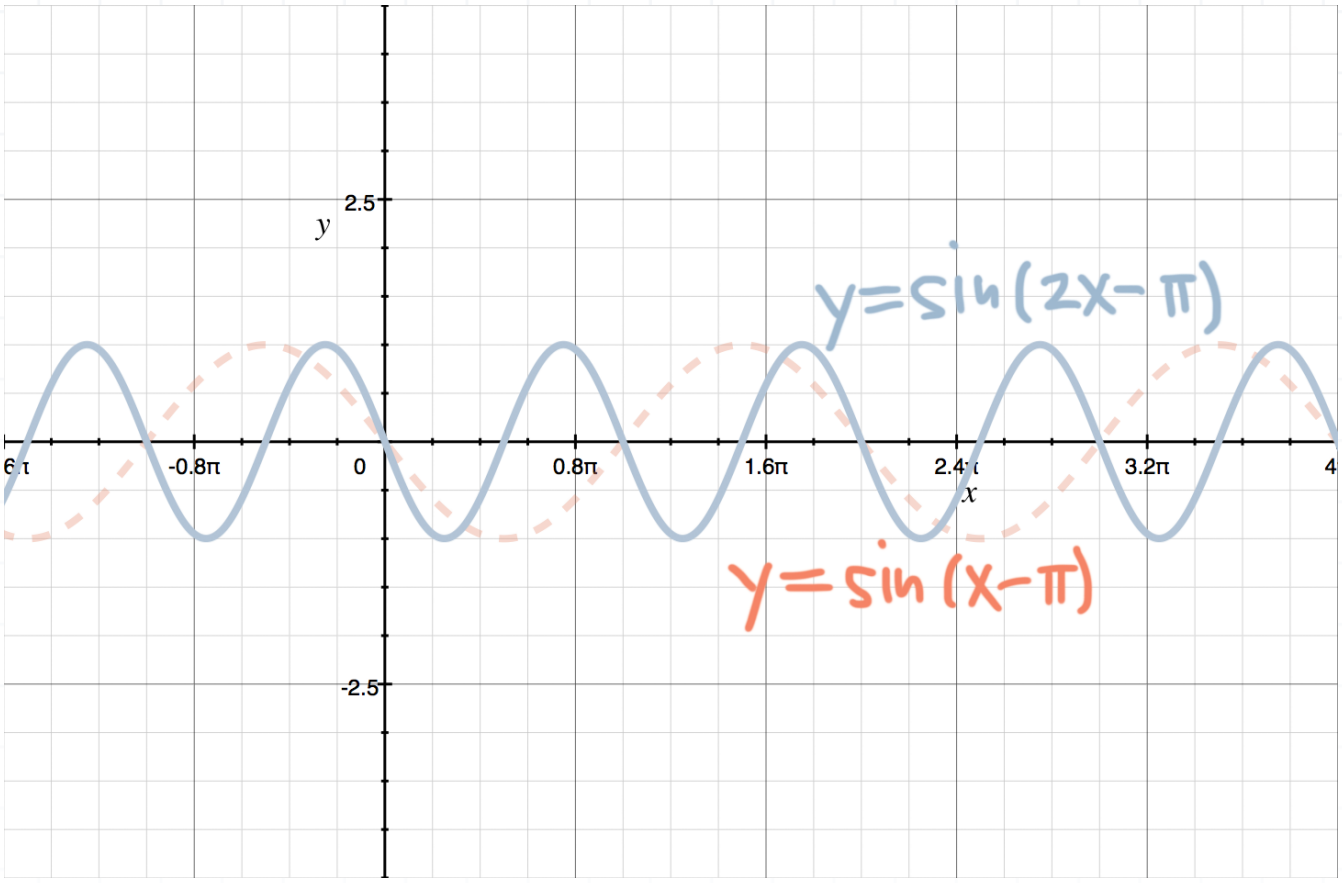

Apply a horizontal shift to the right by 3π by subtracting 3π from the argument.

$$f_2(x) = f_1(x - 3\pi)$$

$$f_2(x) = \sin\left(2(x - 3\pi) - \pi\right)$$

$$f_2(x) = \sin(2x - 7\pi)$$

Since 2π is the period of a sine function, we can simplify $f_2(x)$ by subtracting a multiple of the period, 6π.

$$f_2(x) = \sin(2x - 7\pi - 6\pi)$$

$$f_2(x) = \sin(2x - \pi)$$

A horizontal shift of 3π doesn't change the function in this case.

Apply a vertical stretch by 5 by multiplying the function by 5.

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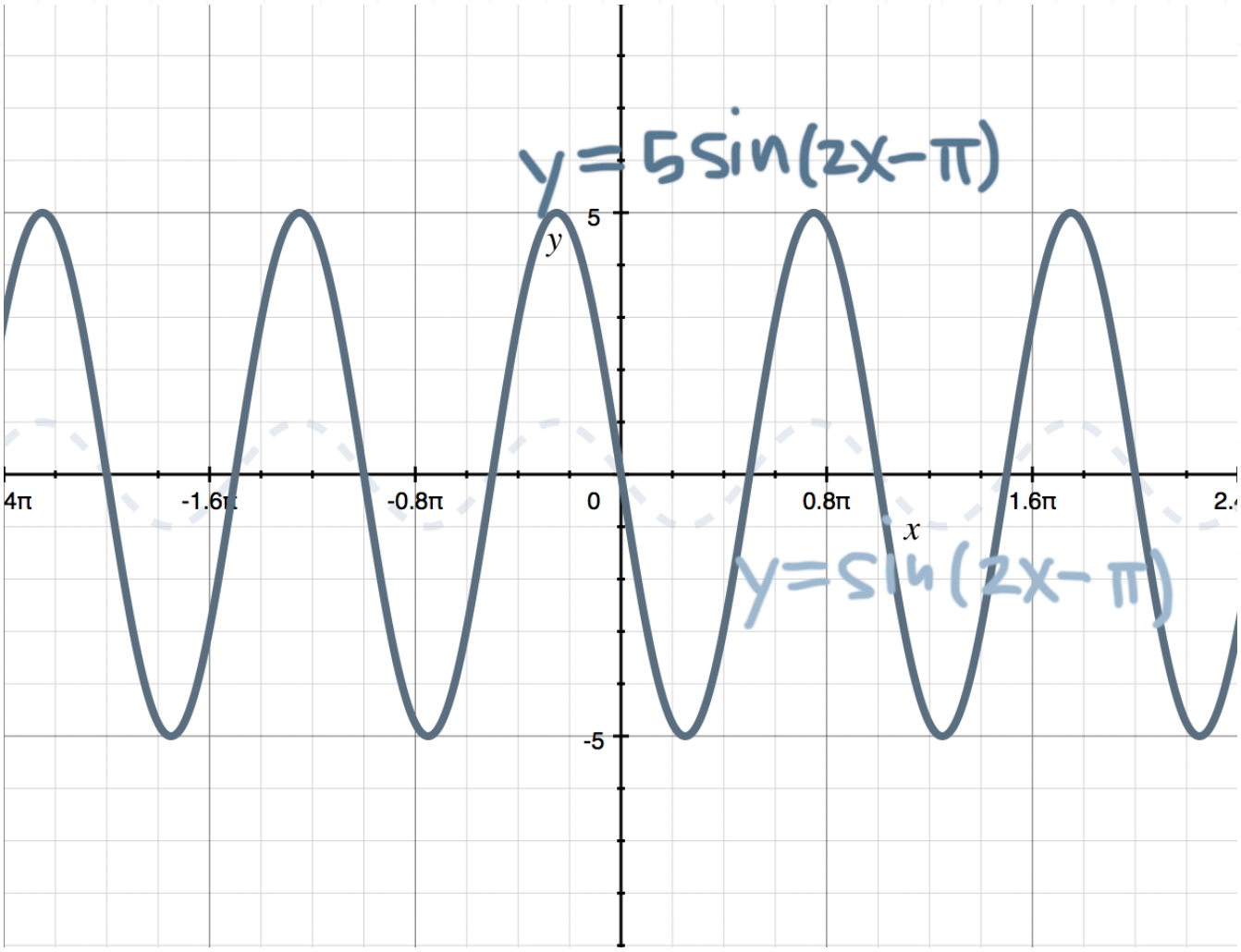
$$f_3(x) = 5f_2(x)$$

$$f_3(x) = 5\sin(2x - \pi)$$

A sketch of the transformation is



Apply a vertical shift down by 2 by subtracting 2 from the function.

$$f_4(x) = f_3(x) - 2$$

$$f_4(x) = 5\sin(2x - \pi) - 2$$

A sketch of the transformation is

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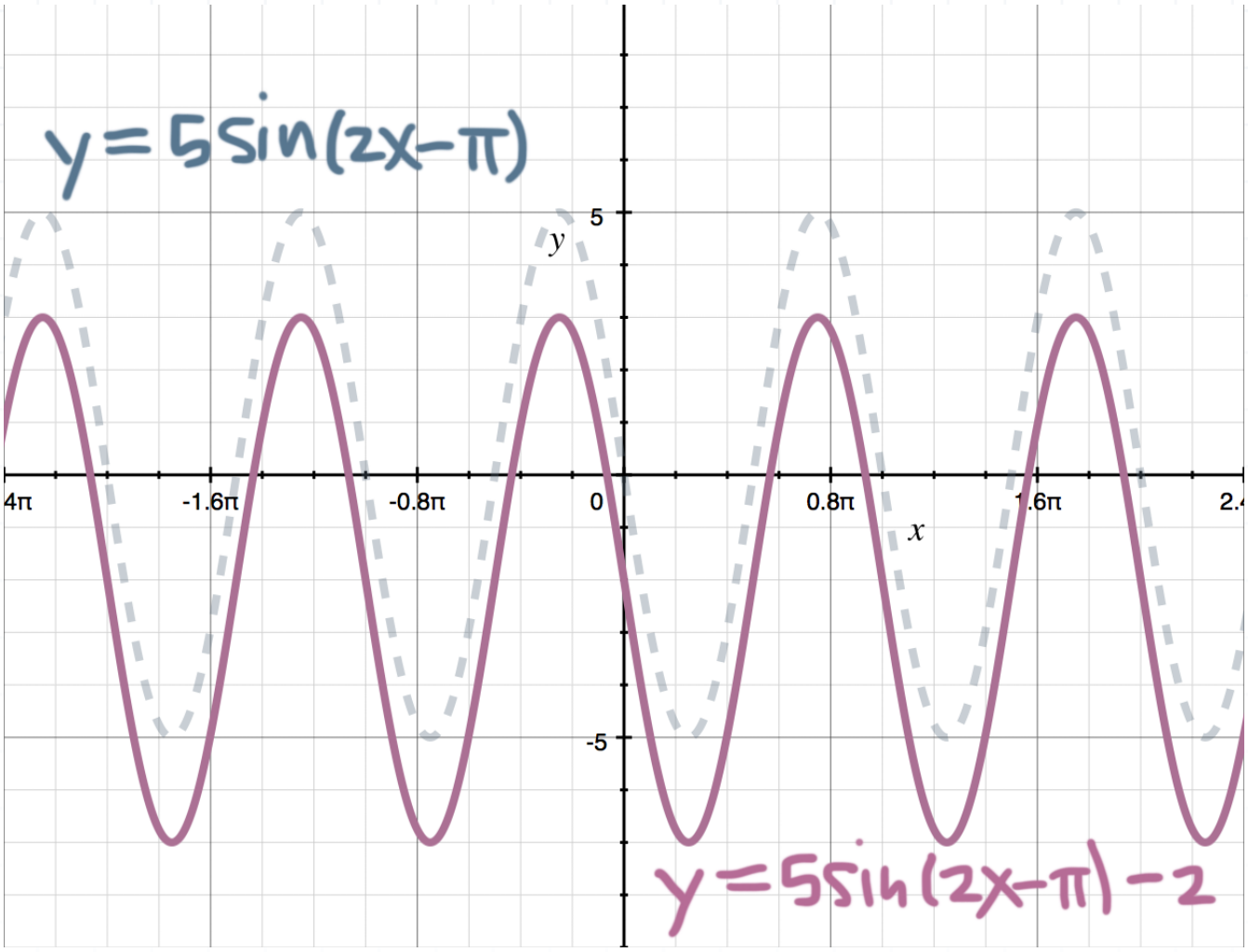

Apply a reflection about the x-axis by multiplying the function by -1.

$$f_5(x) = -f_4(x)$$

$$f_5(x) = -\left(5\sin(2x-\pi) - 2\right)$$

$$f_5(x) = -5\sin(2x-\pi) + 2$$

A sketch of the transformation is

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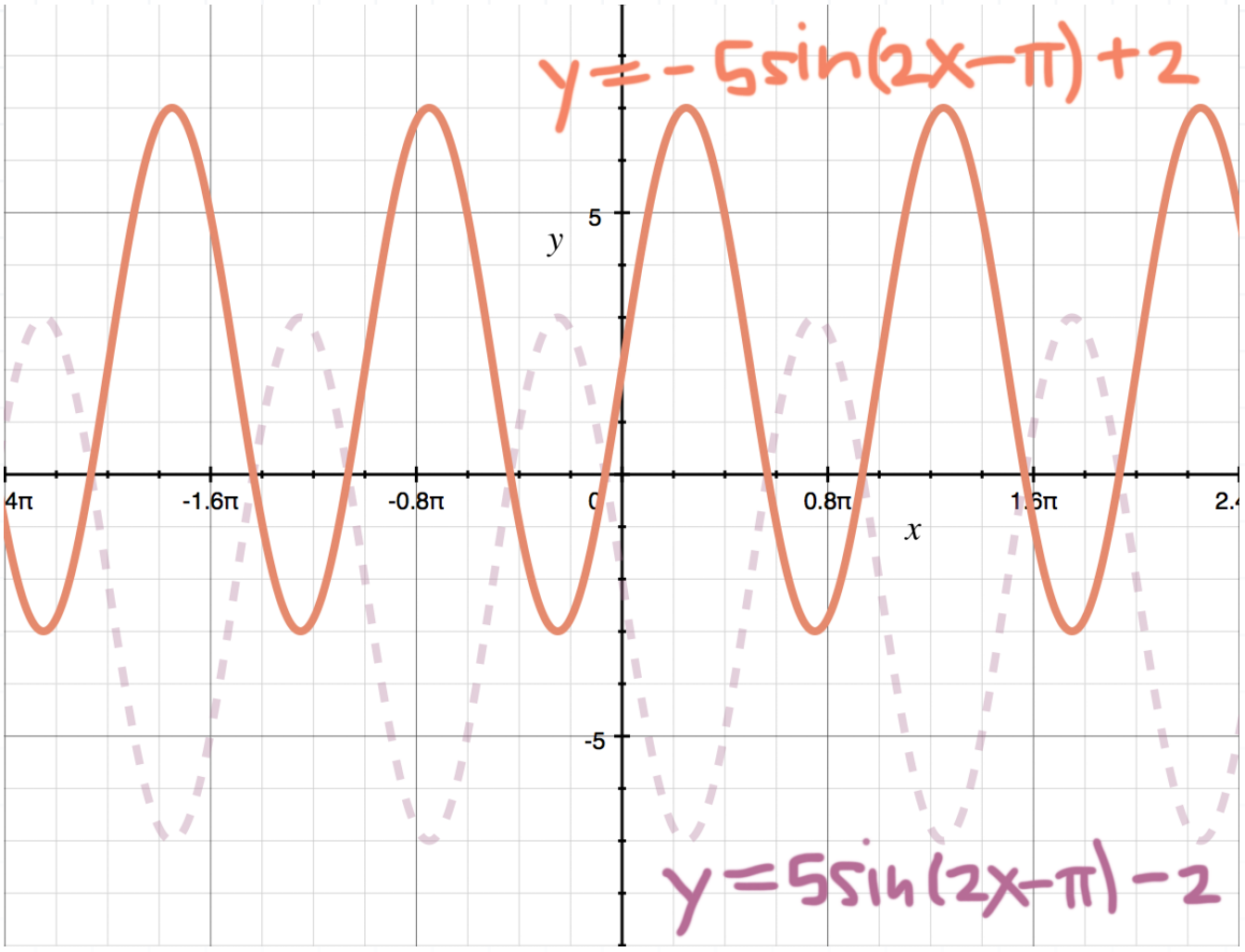

■ 3. Find the set of angles x where the function reaches its maximum value.

$$f(x) = 3\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

Solution:

Factor out $1/2$ inside the parentheses.

$$f(x) = 3\sin\left(\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right)$$

We know that a sine function $g(x) = \sin x$ reaches its maximum value of 1 at

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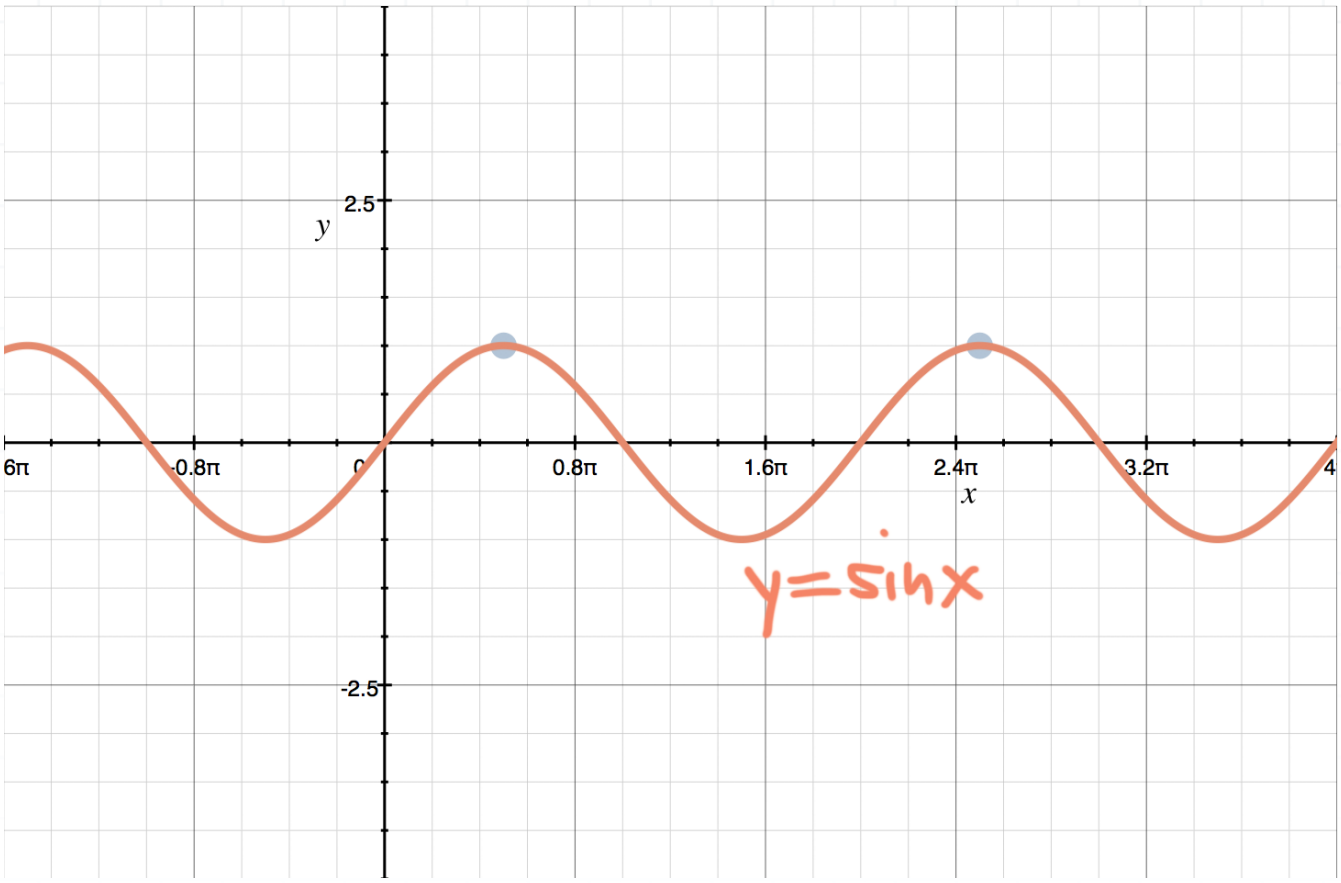
$\frac{\pi}{2} + 2\pi k$, where k is any integer

The points where $\sin x = 1$ look like this:



We can use a sequence of transformations to get the function $f(x)$ from $g(x)$, at the same time making the same horizontal transformations to the points $(\pi/2) + 2\pi k$ (vertical transformations don't change the x-coordinates of the maximum points).

The initial function is

$$g_1(x) = \sin x$$

Apply a horizontal stretch by a factor of 2 by dividing the argument by 2.

$$g_2(x) = g_1\left(\frac{x}{2}\right) = \sin\left(\frac{x}{2}\right)$$

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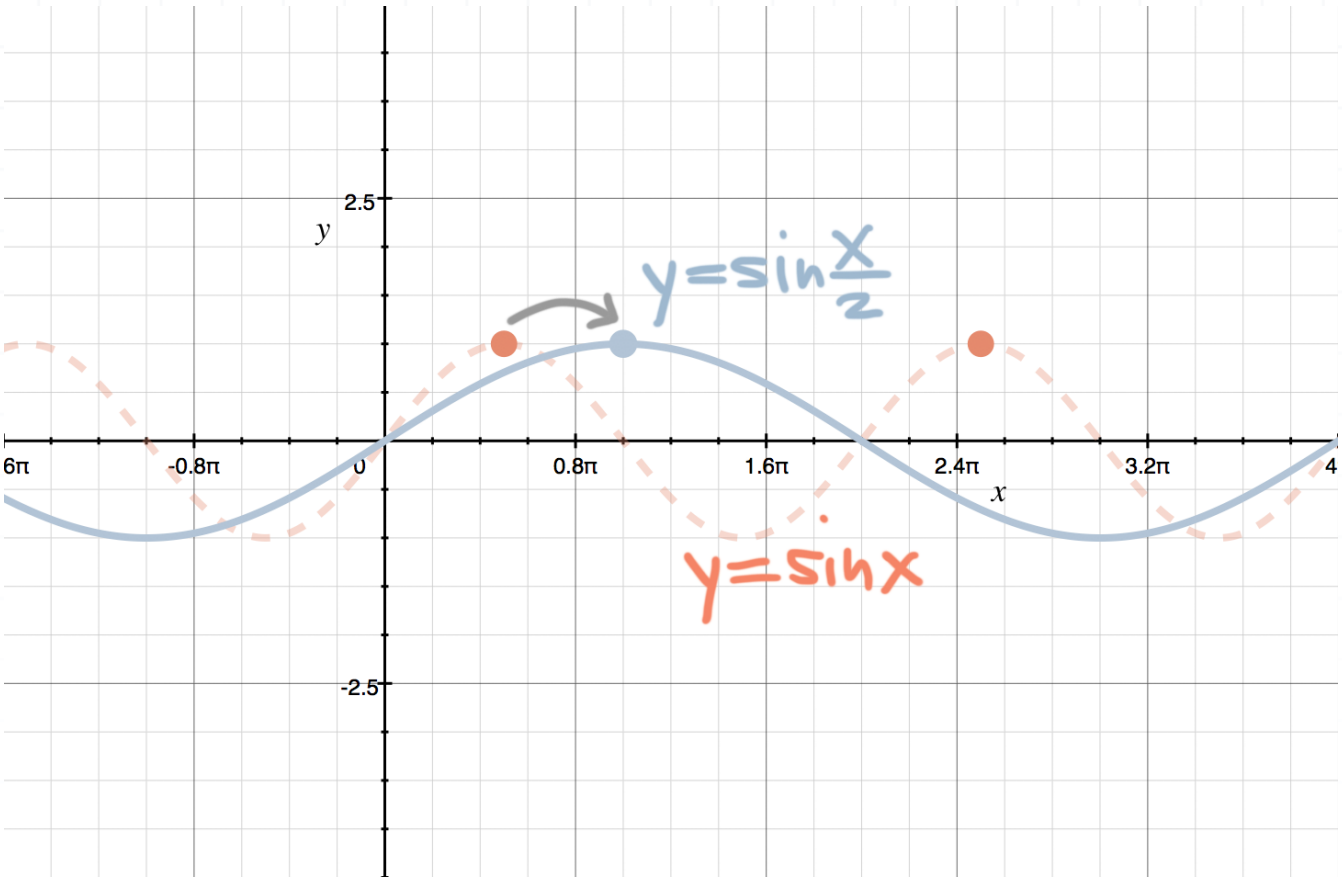
Stretch the maximums by multiplying their values by 2.

$$2\left(\frac{\pi}{2}+2\pi k\right)=\pi+4\pi k$$

A sketch of the transformation is



Apply a horizontal shift to the left by $\pi/2$ by adding $\pi/2$ to the argument.

$$g_3(x)=g_2\left(x+\frac{\pi}{2}\right)$$

$$g_3(x)=\sin\left(\frac{1}{2}\left(x+\frac{\pi}{2}\right)\right)$$

$$g_3(x)=\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

Shift the maximums to the left by subtracting $\pi/2$.

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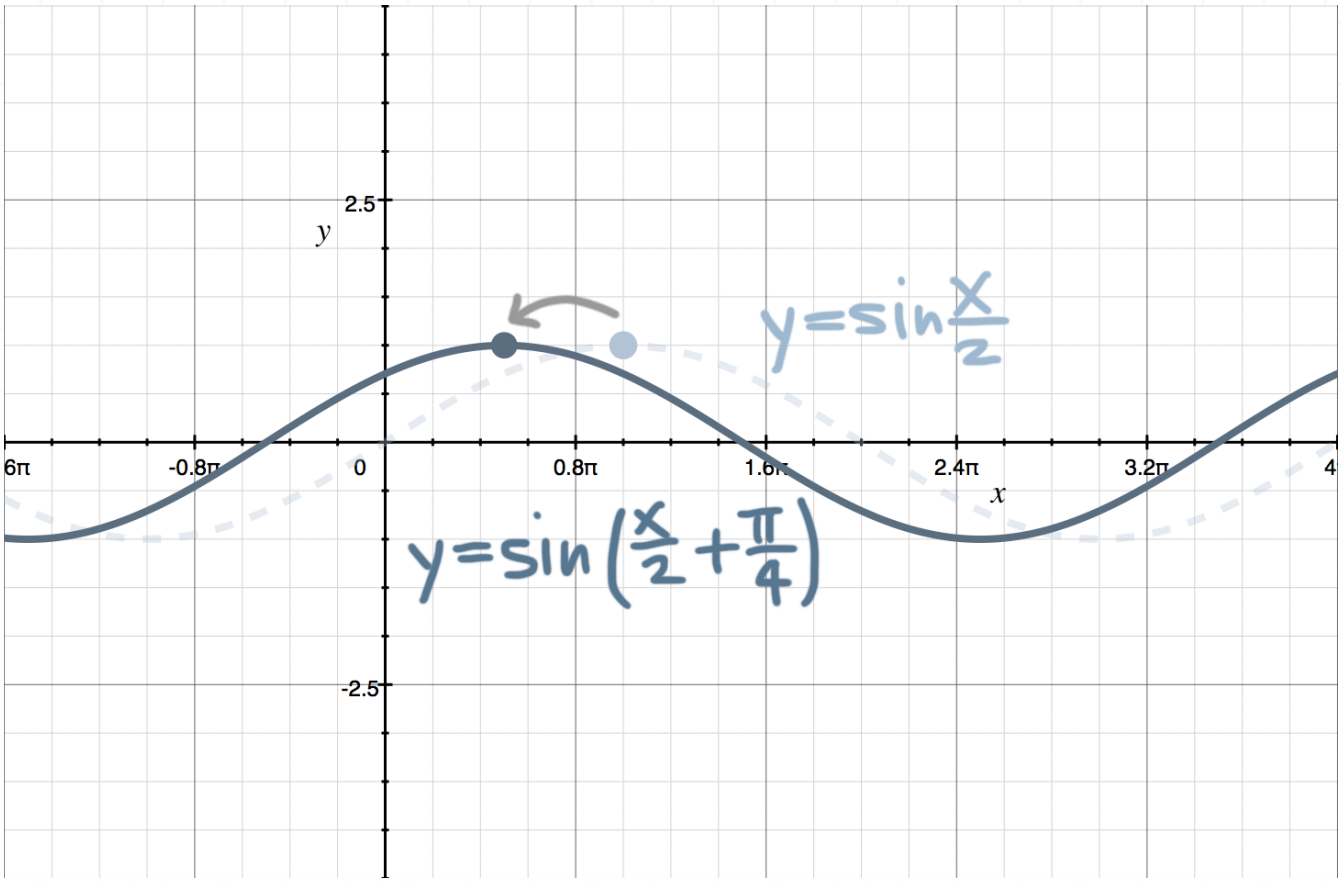
$$\pi + 4\pi k - \frac{\pi}{2} = \frac{\pi}{2} + 4\pi k$$

A sketch of the transformation is



Apply a vertical stretch by a factor of 3 by multiplying the function by 3.

$$g_4(x) = 3\,g_3(x) = 3\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

Since a vertical stretch doesn't affect the x-coordinates of the maximum points, we don't change them.

$\frac{\pi}{2} + 4\pi k$, where k is any integer

A sketch of the transformation is

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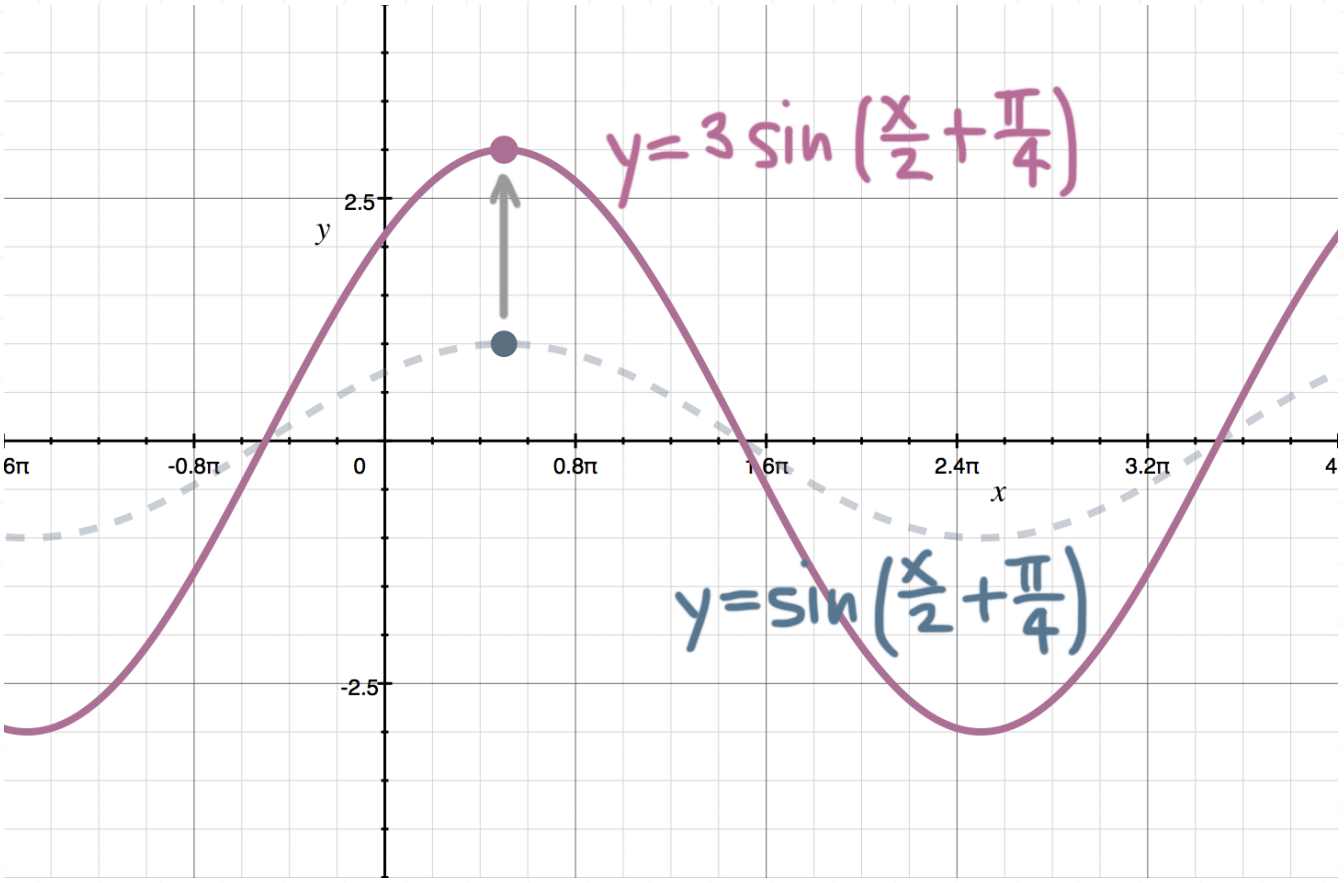

Since $g_4(x) = f(x)$, the set of points above is the solution of the problem. In set notation, the set of angles where the function has its maximum value is

$$\left\{ \frac{\pi}{2} + 4\pi k : \ k \in \mathbb{Z} \right\}$$

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GRAPHING COMBINATIONS OF FUNCTIONS

■ 1. What is the period of the function?

$$f(x) = 2\sin\left(\frac{2x}{3}\right)\left(\cos(x-\pi) + 4\sin\left(\frac{x}{4}\right)\right)$$

Solution:

The period of the sum or product of multiple trig functions is equal to the least common multiple of their periods. If

$$g(x) = \cos(x-\pi) + 4\sin\left(\frac{x}{4}\right)$$

then the period of the cosine part is

$$\text{period}_1 = \frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$$

and the period of the sine part is

$$\text{period}_2 = \frac{2\pi}{b} = \frac{2\pi}{\frac{1}{4}} = 8\pi$$

Since the least common multiple of 2π and 8π is 8π, the period of $g(x)$ is 8π.

Now, consider the function

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$$f(x) = 2\sin\left(\frac{2x}{3}\right) \cdot g(x)$$

It's period is

$$\text{period}_3 = \frac{2\pi}{b} = \frac{2\pi}{\frac{2}{3}} = 3\pi$$

So the period of $f(x)$ is the least common multiple of 3π and 8π, which is 24π.

2. Find the zeros of the function.

$$f(x) = \frac{\sin(\pi x - \pi)}{\sin\left(\frac{\pi x}{2} + 3\pi\right)}$$

Solution:

A fraction will equal 0 when the numerator is 0 and the denominator is not 0. So we need to

(1) find the zeros of the numerator by solving $\sin(\pi x - \pi) = 0$, but

(2) exclude from the answer the solutions of $\sin\left(\frac{\pi x}{2} + 3\pi\right) = 0$.

First we need to solve

$$\sin(\pi x - \pi) = 0$$

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The equation $\sin\theta = 0$ is true when

$\theta = \pi n$ where n is any integer

Plug in $\theta = \pi x - \pi$.

$$\pi x - \pi = \pi n$$

$$\pi x = \pi n + \pi$$

The π term on the right side is already included in πn, so this simplifies to

$$\pi x = \pi n$$

$x = n$ where n is any integer

So the solution set is $\mathbb{Z}$, the set of all integers. Then let's check if any integers make the denominator 0. Think about odd and even integers individually.

If x is an odd integer, then there exists some integer t, such that $x = 2t + 1$.

$$\frac{\pi x}{2} + 3\pi = \frac{\pi\,(2t+1)}{2} + 3\pi = \frac{\pi}{2} + \pi t + 3\pi = \frac{\pi}{2} + \pi(t+3)$$

The denominator becomes

$$\sin\left(\frac{\pi x}{2} + 3\pi\right) = \sin\left(\frac{\pi}{2} + \pi(t+3)\right)$$

which is not equal to 0 for any t. Since we don't have to exclude any odd integers, the zeros of the function $f(x)$ are all odd integer numbers.

If x is an even integer, then there exists some integer t, such that $x = 2t$.

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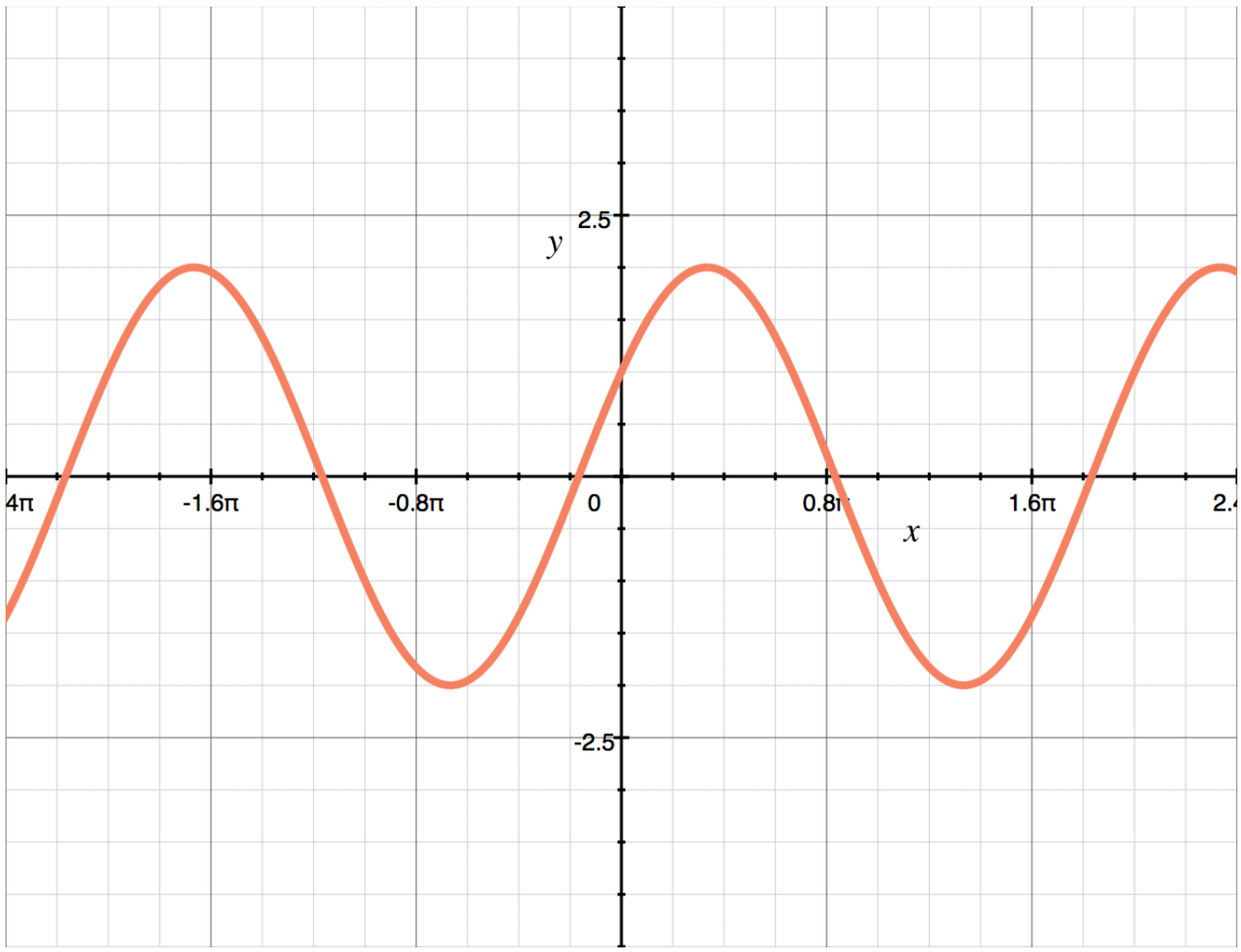
$$\frac{\pi x}{2} + 3\pi = \frac{\pi\,(2t)}{2} + 3\pi = \pi t + 3\pi = \pi(t+3)$$

The denominator becomes

$$\sin\left(\frac{\pi x}{2} + 3\pi\right) = \sin(\pi(t+3))$$

which is equal to 0 for any t. So all even numbers should be excluded from the solution set. Which means the zeros of the function are all odd integers.

■ 3. We know that the graph of $f(x) = \sqrt{3}\sin x + \cos x$ sketched out in the graph below is identical to the graph of some sine function $g(x) = a\sin(bx + c)$. Find $g(x)$.



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Solution:

If we're going for

$$g(x) = a\sin(bx + c)$$

first find b using the period of $f(x)$. The period of the sum of multiple trig functions is equal to the least common multiple of their periods.

Since the period of $\sqrt{3}\sin x$ is 2π, and the period of $\cos x$ is 2π, the period of their sum is also 2π.

Since the period of $g(x) = 2\pi$, the parameter b is equal to 1. Plug $b = 1$ into the equation for $g(x)$.

$$g(x) = a\sin(x + c)$$

We can use the fact that the graph in the image has an amplitude of 2 to say that the parameter a can accept values -2 or 2. Start by considering $a = 2$.

$$g(x) = 2\sin(x + c)$$

We know that $g(0) = f(0)$, so

$$2\sin(0 + c) = \sqrt{3}\sin 0 + \cos 0$$

$$2\sin(0 + c) = 1$$

$$\sin c = \frac{1}{2}$$

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$$c = \frac{\pi}{6}$$

Since we need only one solution (even though there are many other possible solutions), we can stop here and write down an equation for $g(x)$.

$$g(x) = 2\sin\left(x + \frac{\pi}{6}\right)$$

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TRANSFORMATIONS ALGEBRAICALLY

■ 1. The function $g(x)$ is made from $f(x)$ by applying a sequence of transformations:

Horizontal shift to the right by 3, then

Horizontal stretching by 2, then

Horizontal shift to the left by 4, then

Vertical compression by 2.

Find the zeros of the function $g(x)$.

$$f(x) = \frac{x^3 - 3x^2}{2x^2 + 5}$$

Solution:

Instead of finding $g(x)$ and its zeros, we can just find the zeros of $f(x)$ and then apply the transformations to these points. This is an easier approach, but it only works if there are no vertical shifts. In this case, we're not making a vertical shift as part of the sequence of transformations, so let's find the zeros of $f(x)$.

$$f(x) = 0$$

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$$\frac{x^3 - 3x^2}{2x^2 + 5} = 0$$

Since $2x^2 + 5x$ is positive for any values of x, we can say

$$x^3 - 3x^2 = 0$$

$$x^2(x - 3) = 0$$

$x = 0$ and $x = 3$

So the zeros of $f(x)$ are the points $(0,0)$ and $(3,0)$. Now we'll apply the sequence of transformations to these points one by one.

To apply the horizontal shift to the right by 3 units, increase the x-coordinate of the point by 3.

$$(0 + 3, 0) = (3,0)$$

$$(3 + 3, 0) = (6,0)$$

To apply the horizontal stretch by 2 units, multiply the x-coordinate of the point by 2.

$$(3 \cdot 2,0) = (6,0)$$

$$(6 \cdot 2,0) = (12,0)$$

To apply the horizontal shift to the left by 4 units, decrease the x-coordinate of the point by 4.

$$(6 - 4,0) = (2,0)$$

$$(12 - 4,0) = (8,0)$$

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To apply the vertical compression by 2 units, divide the y-coordinate of the point by 2. Since the vertical compression doesn't change the points on the x-axis, the points are the same, and the zeros of $g(x)$ are

$$(2,0)$$

$$(8,0)$$

■ 2. Find the sequence of transformations needed to transform $f(x)$ into the basic parabola $y = x^2$.

$$f(x) = 2x^2 - 8x + 3$$

Solution:

Start by completing the square.

$$f(x) = 2(x^2 - 4x) + 3$$

$$f(x) = 2\left[(x^2 - 4x + 4) - 4\right] + 3$$

$$f(x) = 2\left[(x - 2)^2 - 4\right] + 3$$

$$f(x) = 2(x - 2)^2 - 8 + 3$$

$$f(x) = 2(x - 2)^2 - 5$$

A sketch of this parabola is

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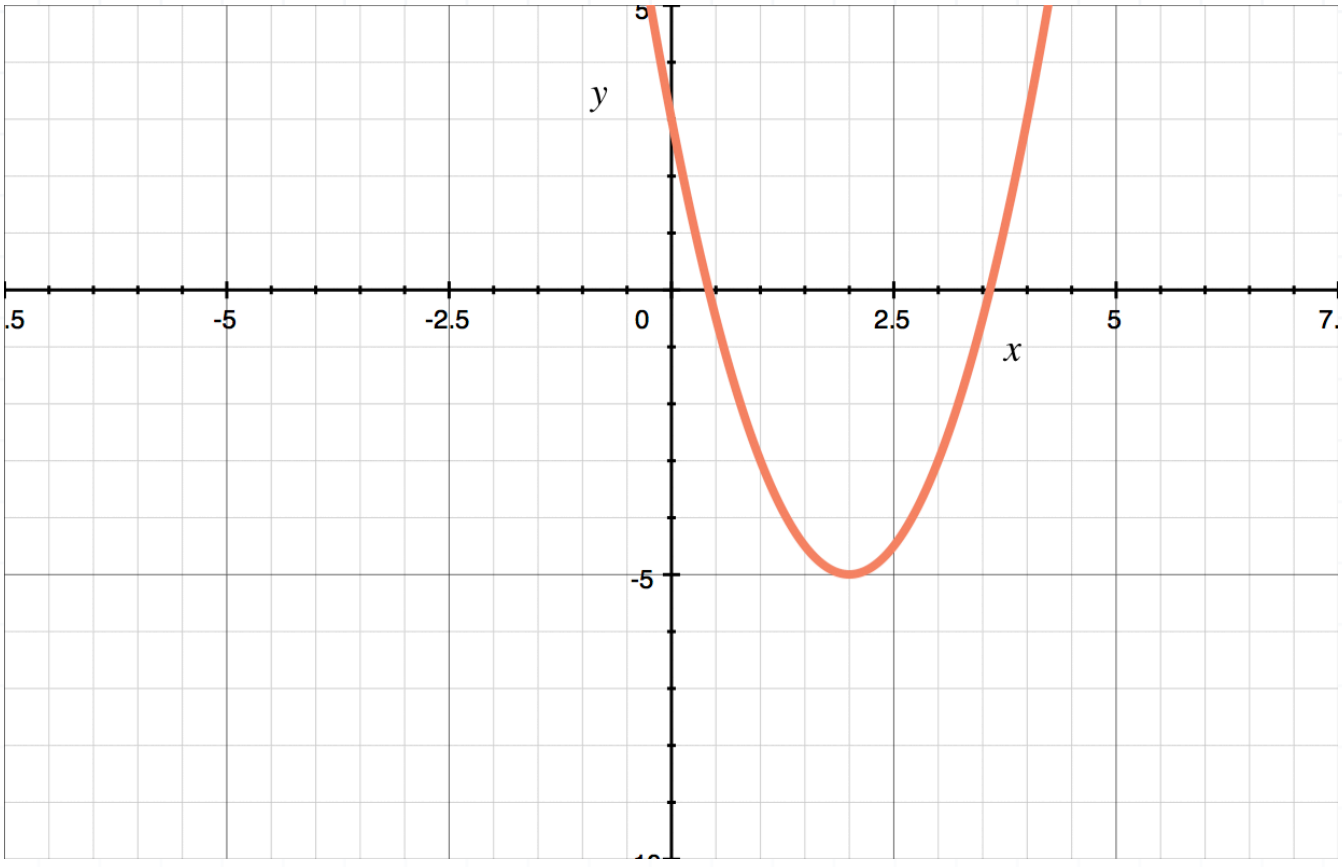

With the equation of the parabola in standard form, we can find the transformations that reduce it to the basic parabola $y = x^2$.

Add 5 to the function to make a vertical shift upward by 5.

$$f_1(x) = f(x) + 5$$

$$f_1(x) = 2(x-2)^2 - 5 + 5$$

$$f_1(x) = 2(x-2)^2$$

Divide the function by 2 to make a vertical compression by 2.

$$f_2(x) = \frac{f_1(x)}{2}$$

$$f_2(x) = \frac{2(x-2)^2}{2}$$

$$f_2(x) = (x-2)^2$$

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Add 2 to the argument to make a horizontal shift by 2.

$$f_3(x) = f_2(x + 2)$$

$$f_3(x) = (x + 2 - 2)^2$$

$$f_3(x) = x^2$$

So we get the parabola $y = x^2$ after three transformations:

Vertical shift upward by 5

Vertical compression by 2

Horizontal shift to the left by 2

■ 3. For $f(x) = -x^2 + 4x$, find its reflection:

Across the x-axis, then

Across the y-axis, then

About the origin, and then

Across the line $y = x$

Solution:

The reflection of $f(x)$ across the x-axis is the function $-f(x)$.

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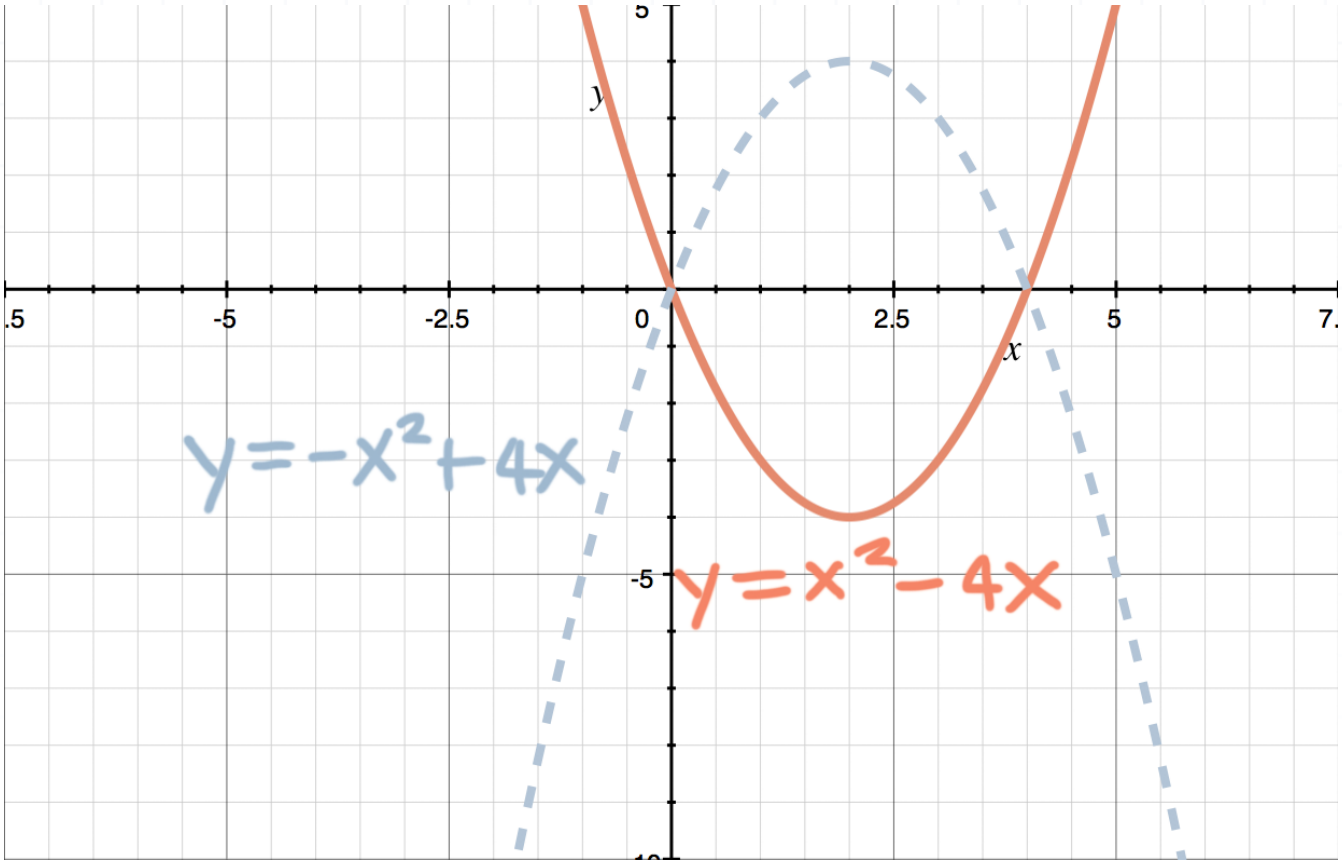
$$f_1(x) = -f(x)$$

$$f_1(x) = -(-x^2 + 4x)$$

$$f_1(x) = x^2 - 4x$$

A sketch of the transformation is



The reflection of $f(x)$ across the y-axis is the function $f(-x)$.

$$f_2(x) = f(-x)$$

$$f_2(x) = -(-x)^2 + 4(-x)$$

$$f_2(x) = -x^2 - 4x$$

A sketch of the transformation is

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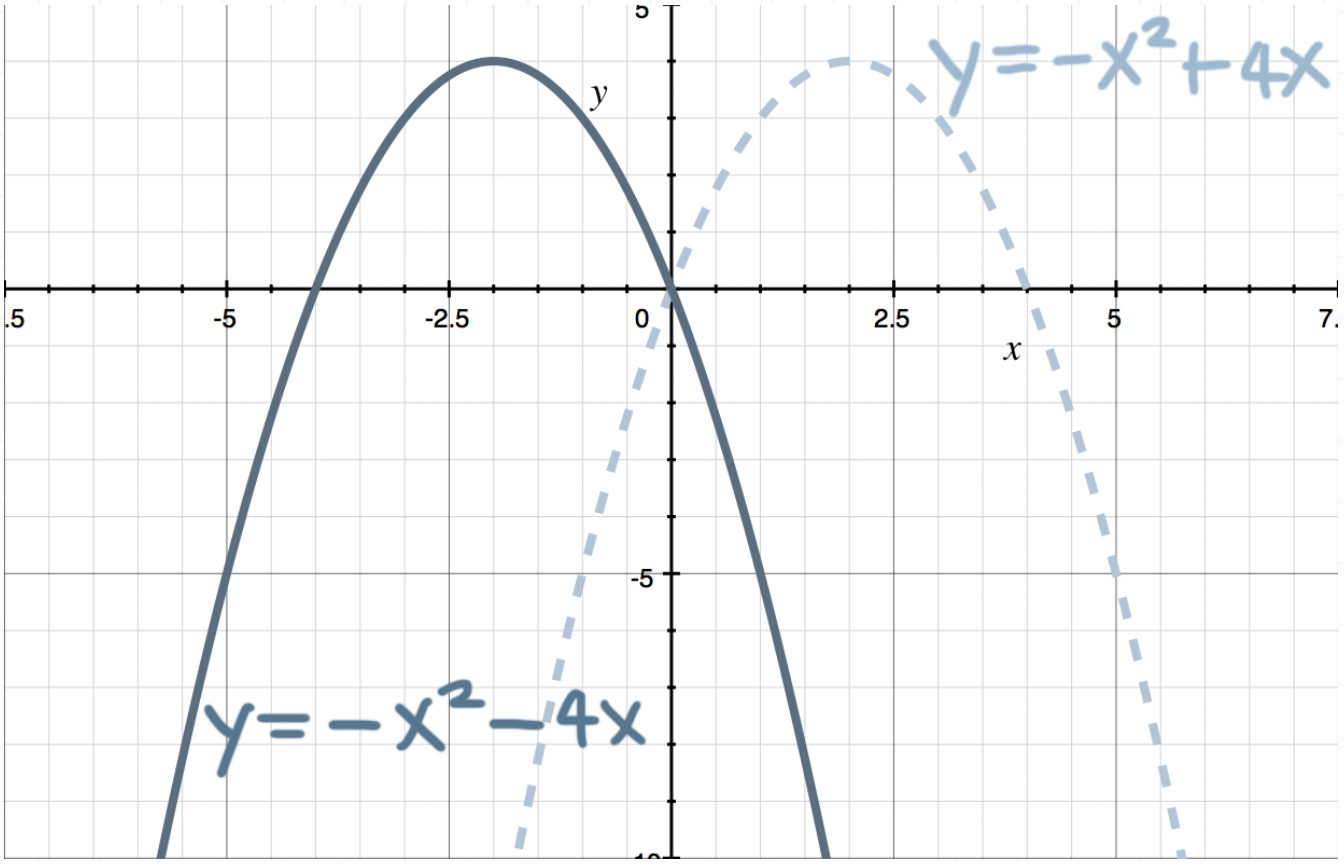

The reflection of $f(x)$ about the origin is the function $-f(-x)$.

$$f_3(x) = -f(-x)$$

$$f_3(x) = -\left(-(-x)^2 + 4(-x)\right)$$

$$f_3(x) = x^2 + 4x$$

A sketch of the transformation is

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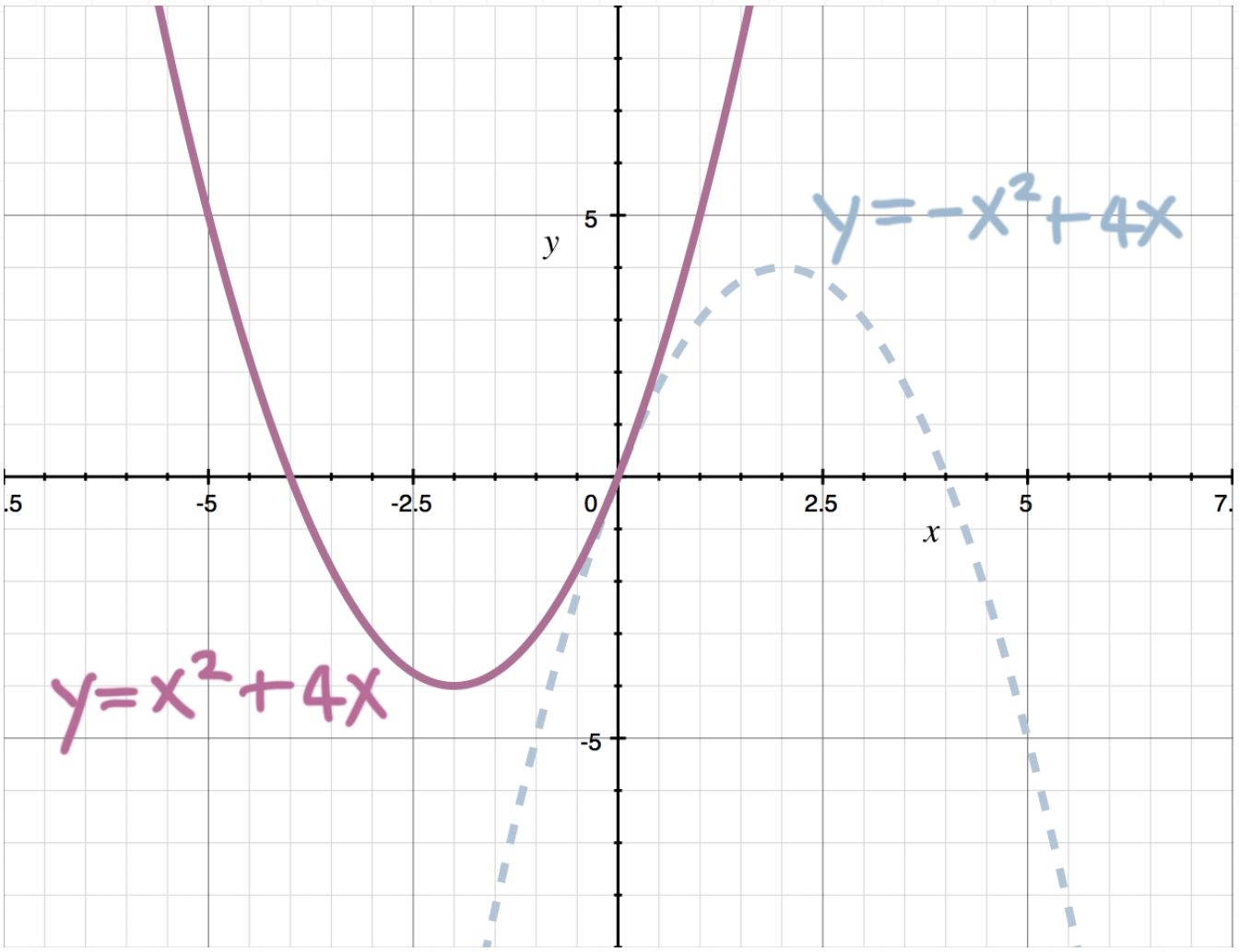

The reflection across the line $y = x$, makes the x- and y-coordinates switch places. So we get

$$y = -x^2 + 4x$$

$$x = -y^2 + 4y$$

$$f(y) = -y^2 + 4y$$

A sketch of the transformation to $f(y) = -y^2 + 4y$ is

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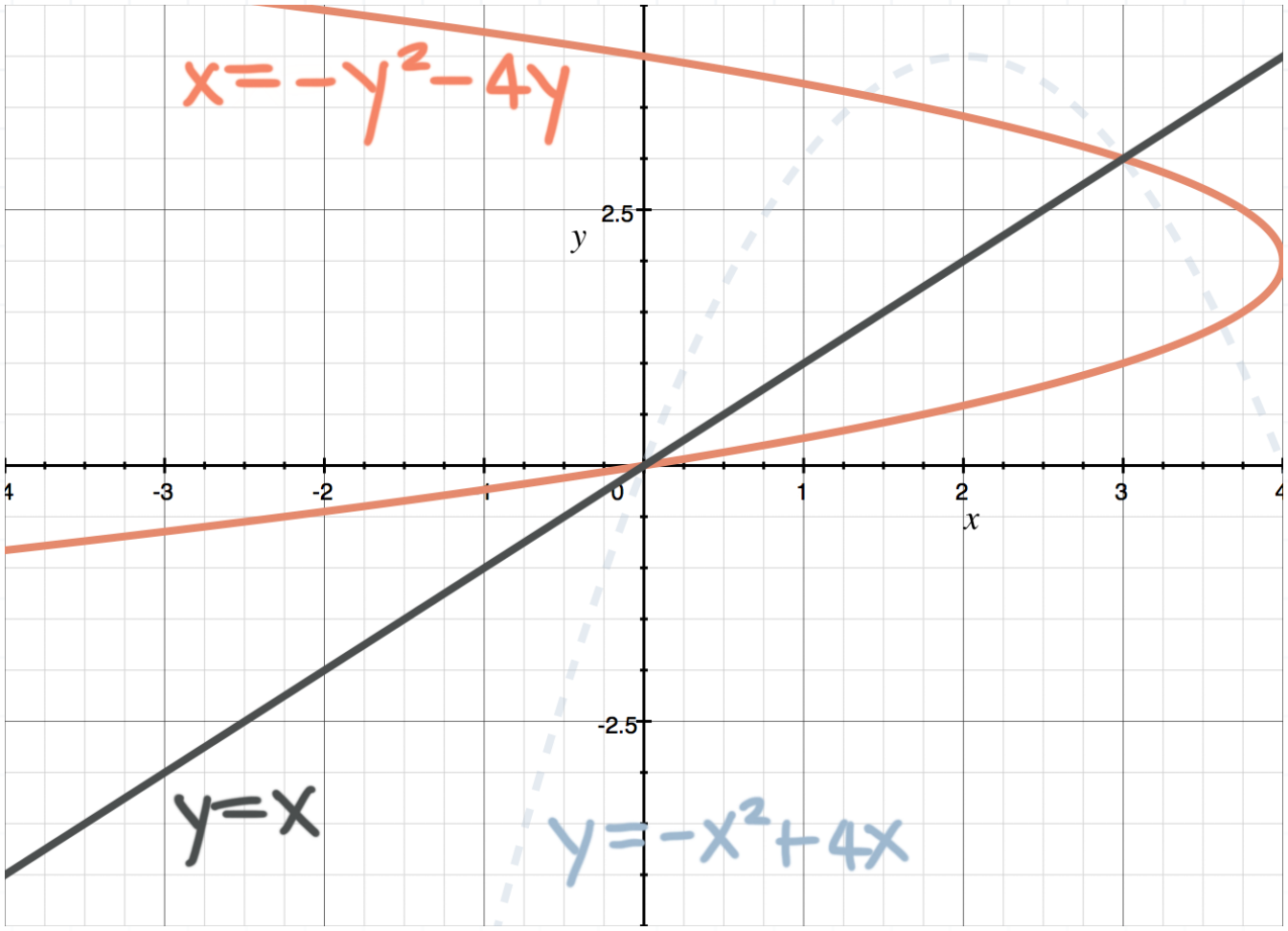

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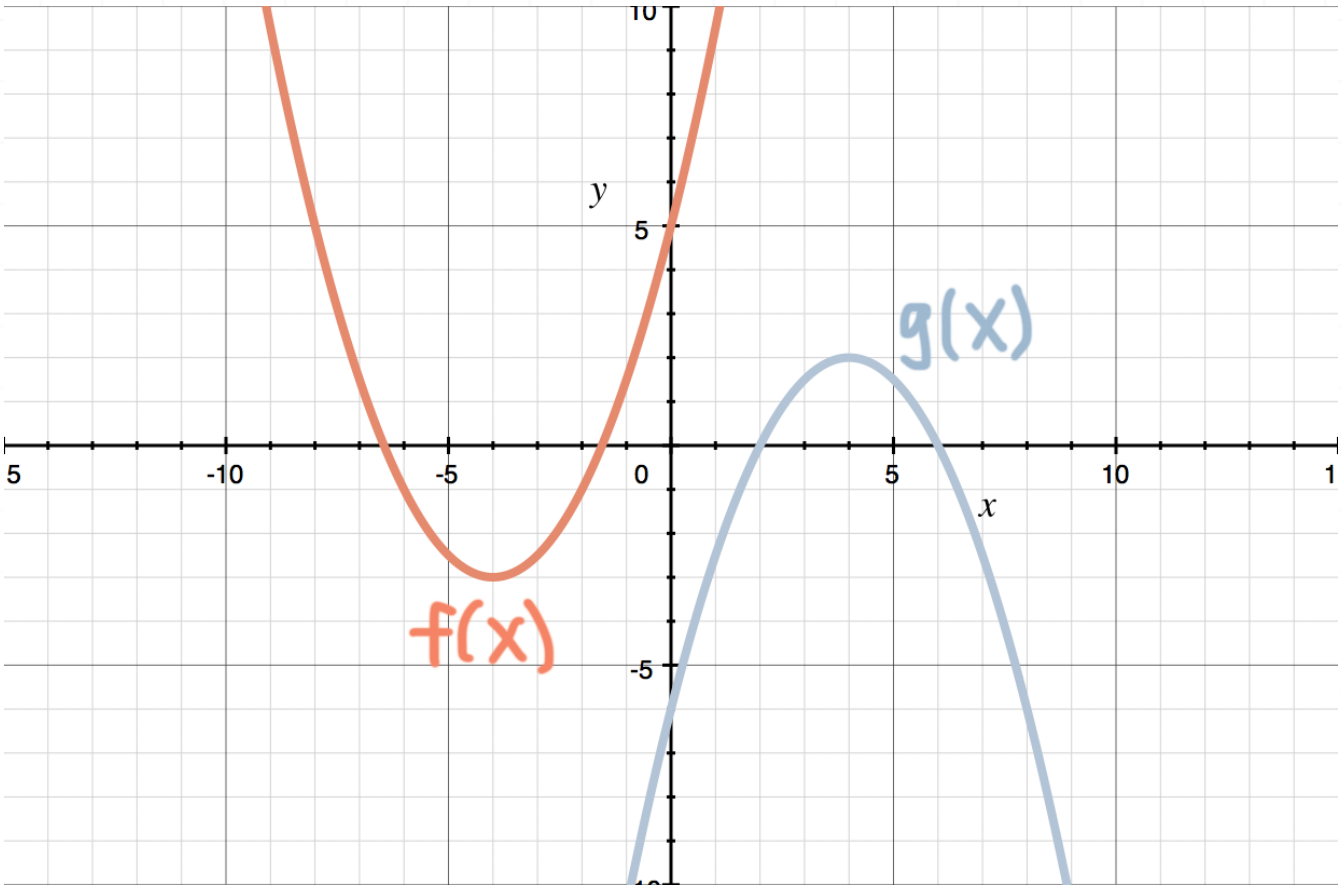
SKETCHING GRAPHS OF TRANSFORMATIONS

■ 1. Find the minimum number of transformations (shifts, compressions, stretching, reflection, about the axes and the origin) required to change $f(x)$ into $g(x)$.



Solution:

There are two ways to solve the problem using only two transformations, and there are no ways to do it using one transformation. So the minimum number of transformation is 2. Let's look at two ways to do it in two steps.

The first example requires these steps:

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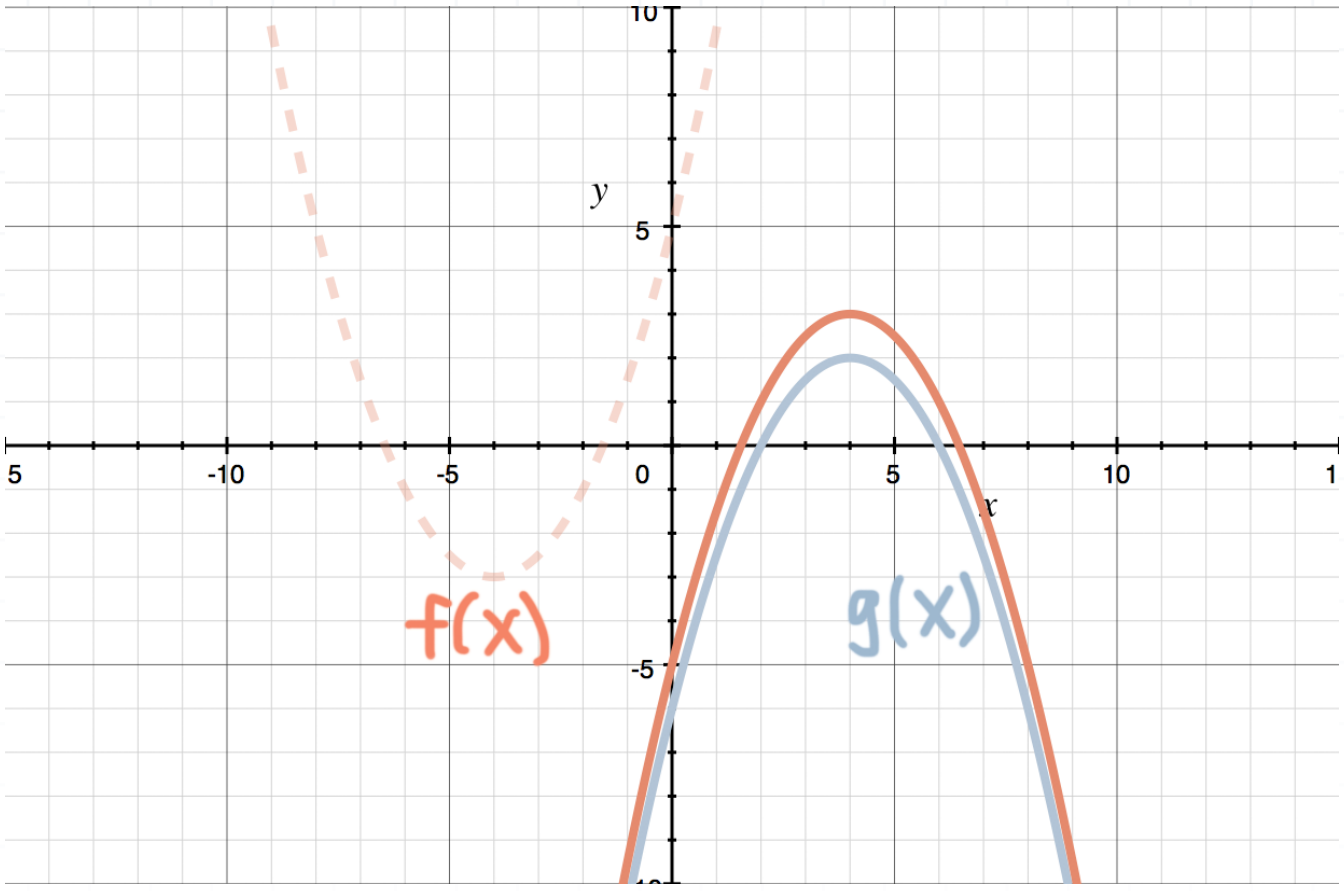
(1) Reflect over the origin to get the concave down parabola with vertex $(4,3)$.



(2) Shift down by 1 to get the parabola with vertex $(4,2)$.

The second example is almost the same as the first, but shifts first and reflects second.

(1) Shift up by 1 to get the parabola with vertex $(-4,-2)$.

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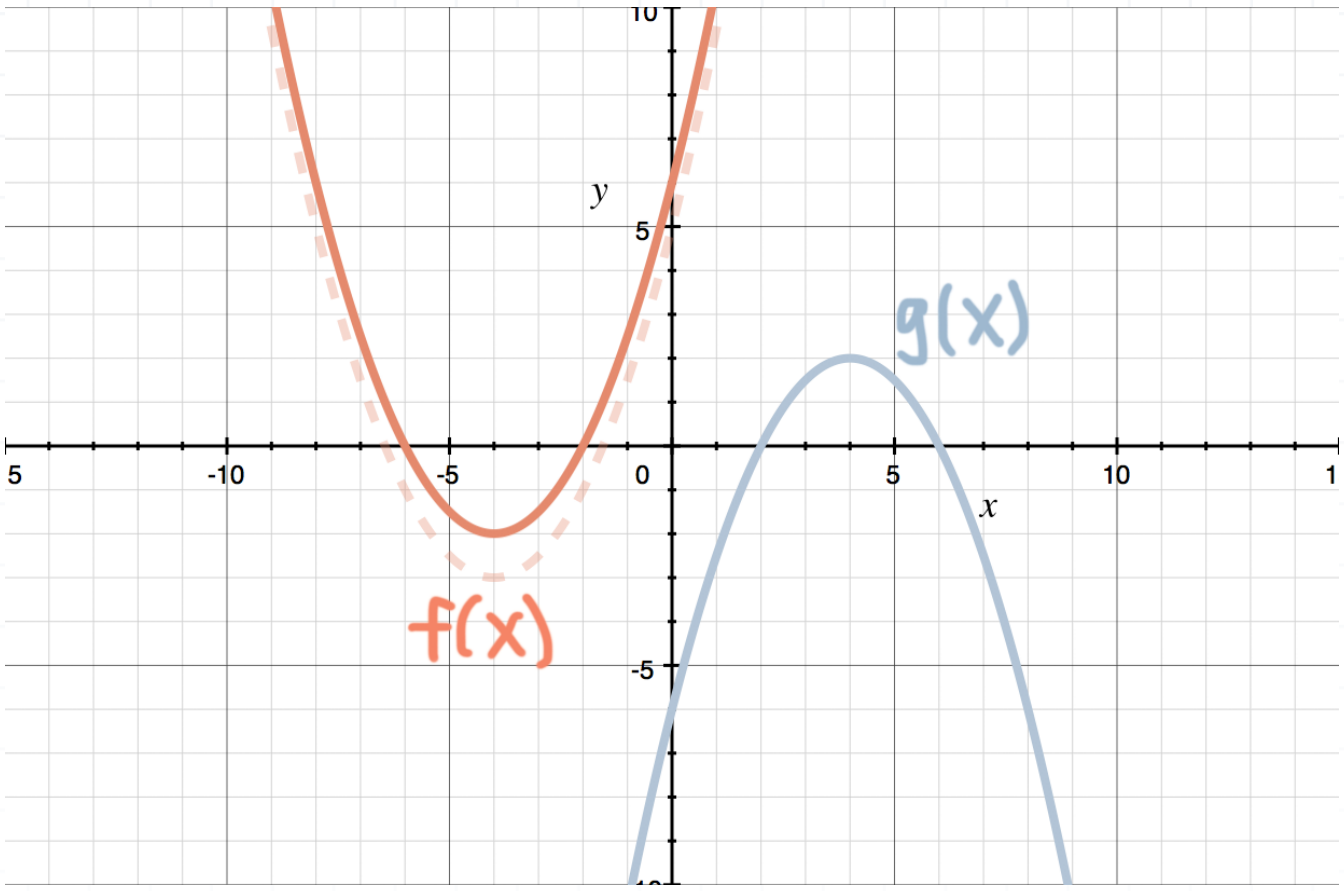

(2) Reflect over the origin to get the concave down parabola with vertex $(4,2)$.

2. Sketch the graph of the piecewise function $f(x)$ shown in the graph after the sequence of transformations:

A vertical compression by 2, then

A horizontal shift to the right by 1, then

A reflection about the y-axis, then

A vertical stretching by 2, and then

A horizontal shift to the left by 1.

The sketch of the function is

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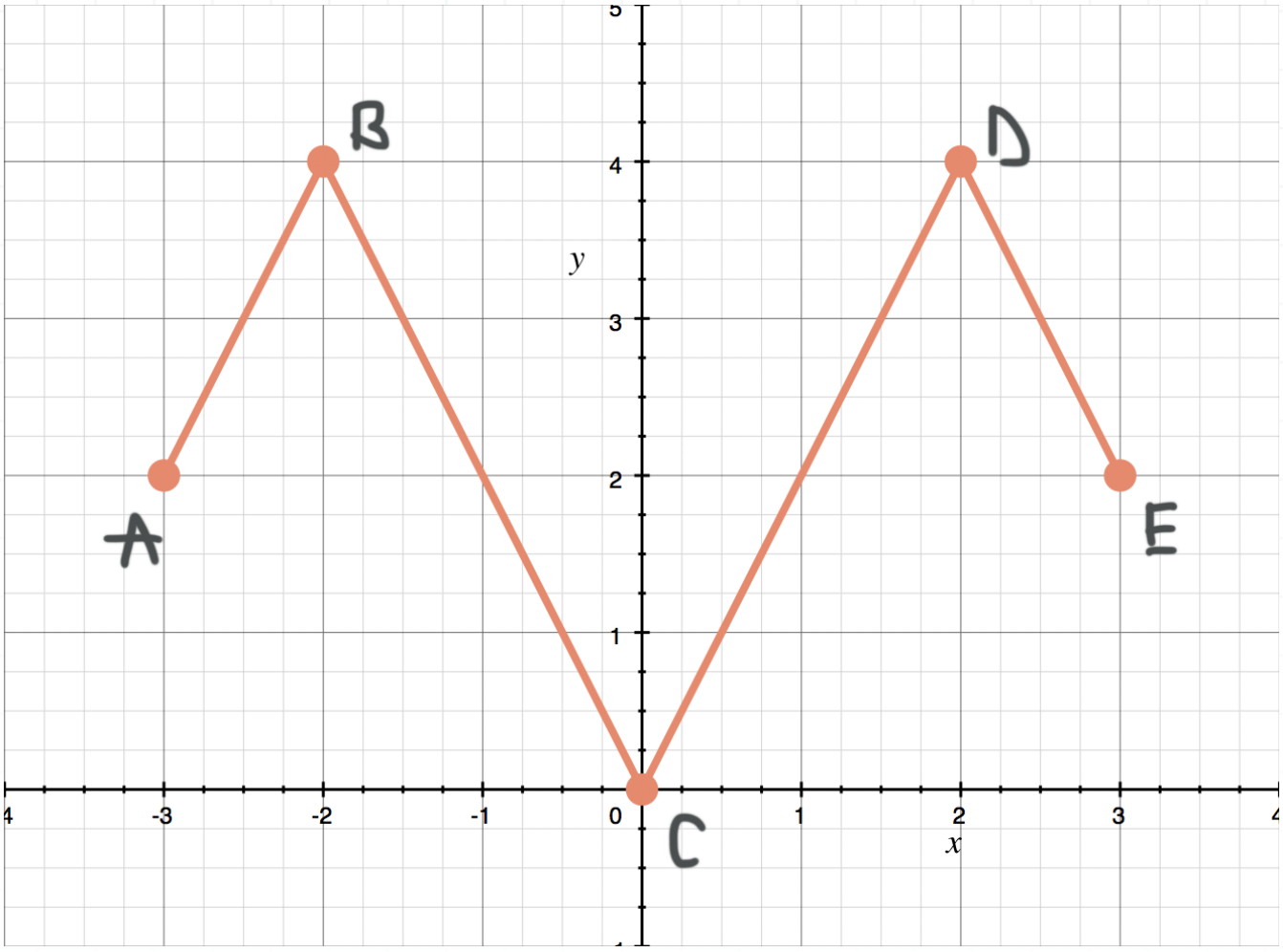

Solution:

Let's make the transformations one by one.

Apply a vertical compression of 2 by dividing the y-coordinates by 2.

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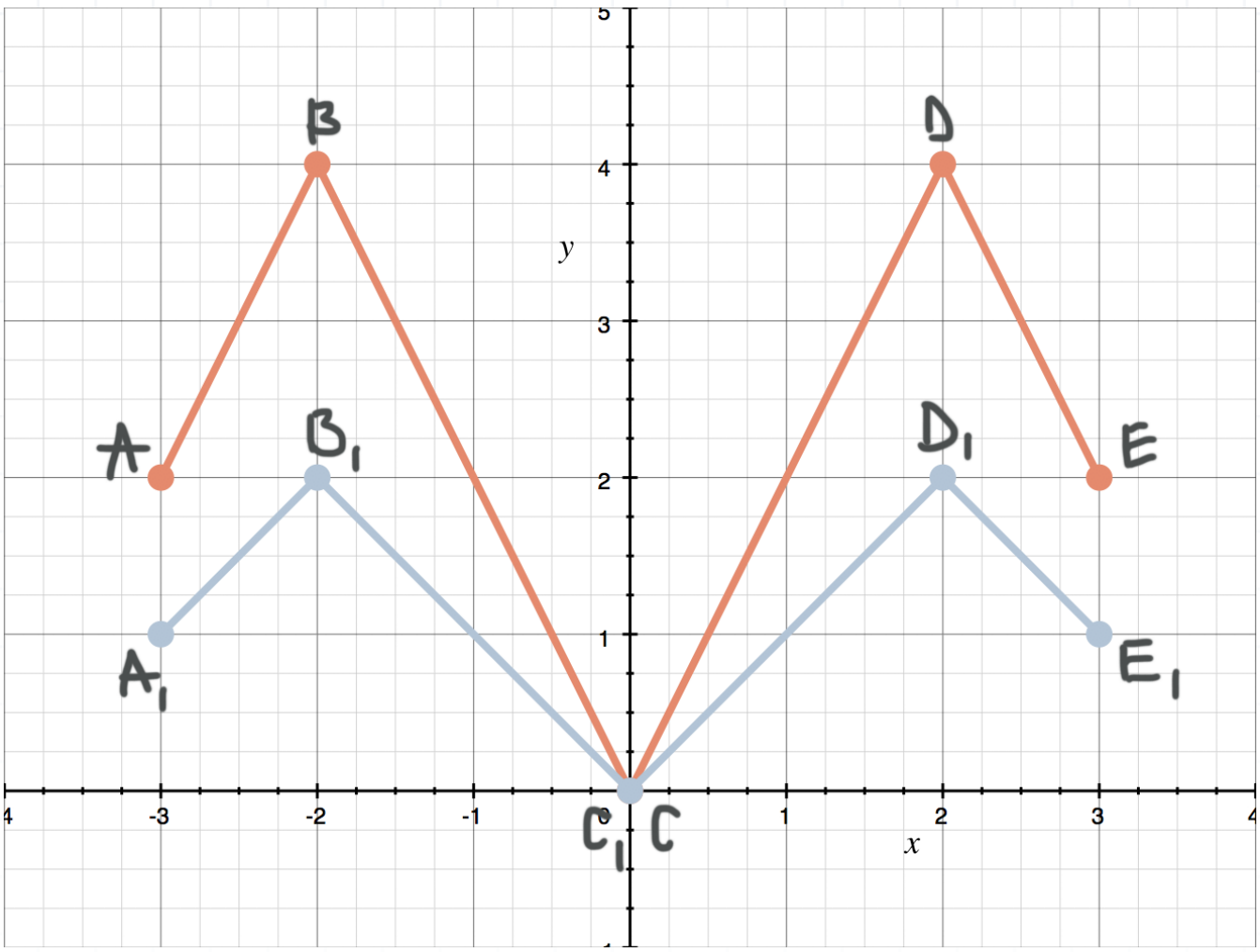

Apply a horizontal shift to the right of 1 by increasing the x-coordinates of the points by 1.

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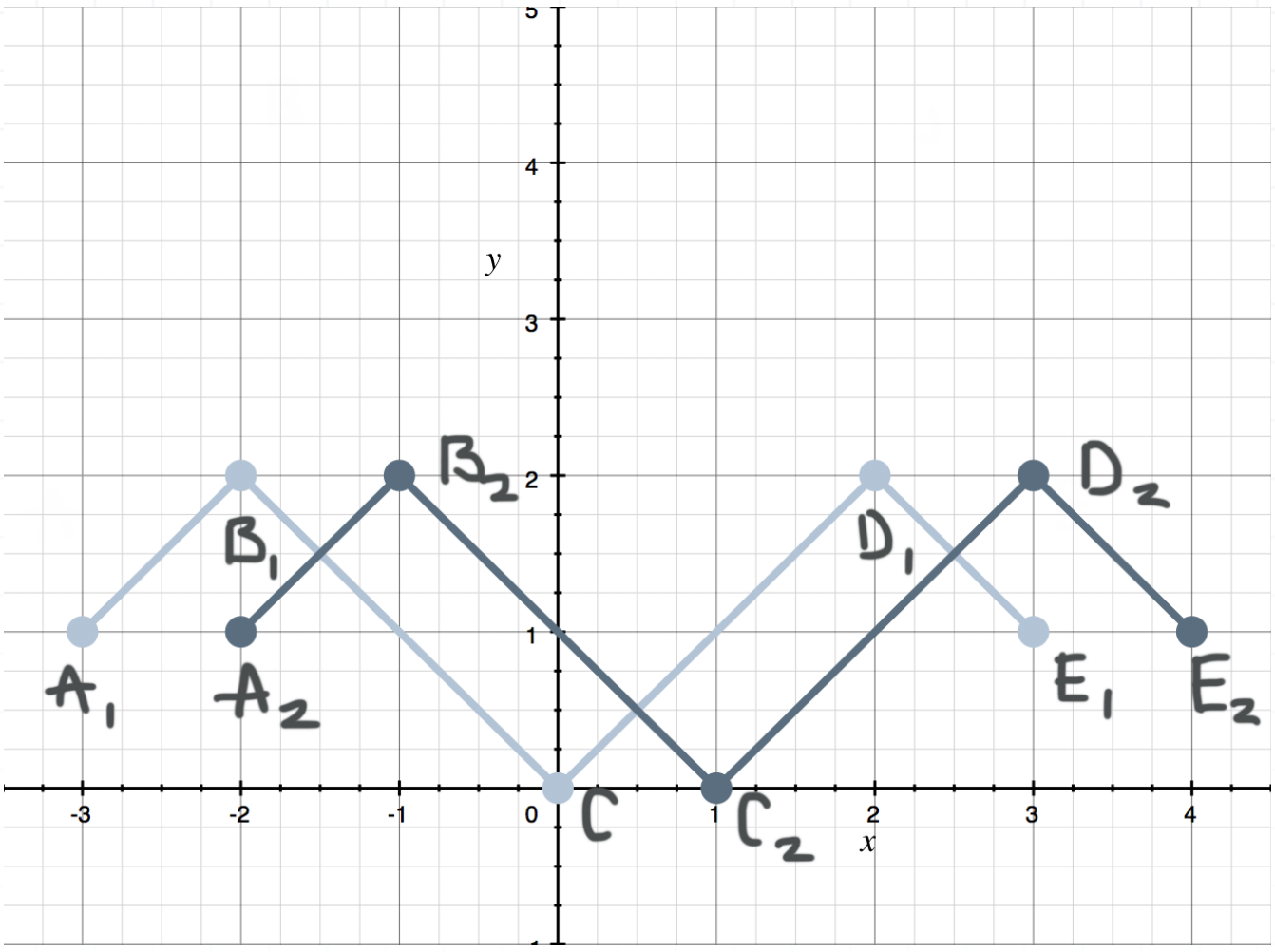
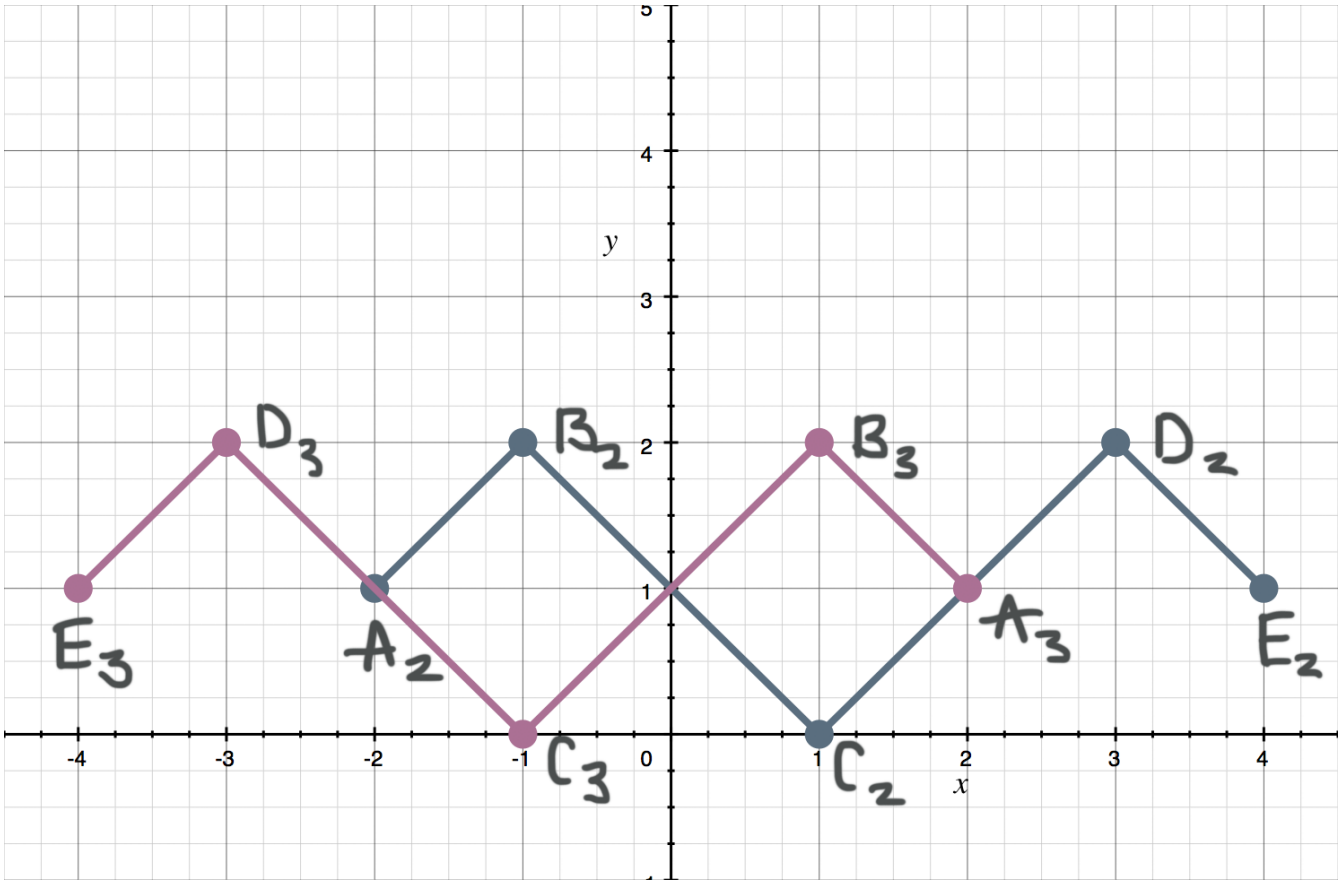


Apply a reflection over the y-axis by changing the x-coordinates of the points into their opposites (the signs are changed).



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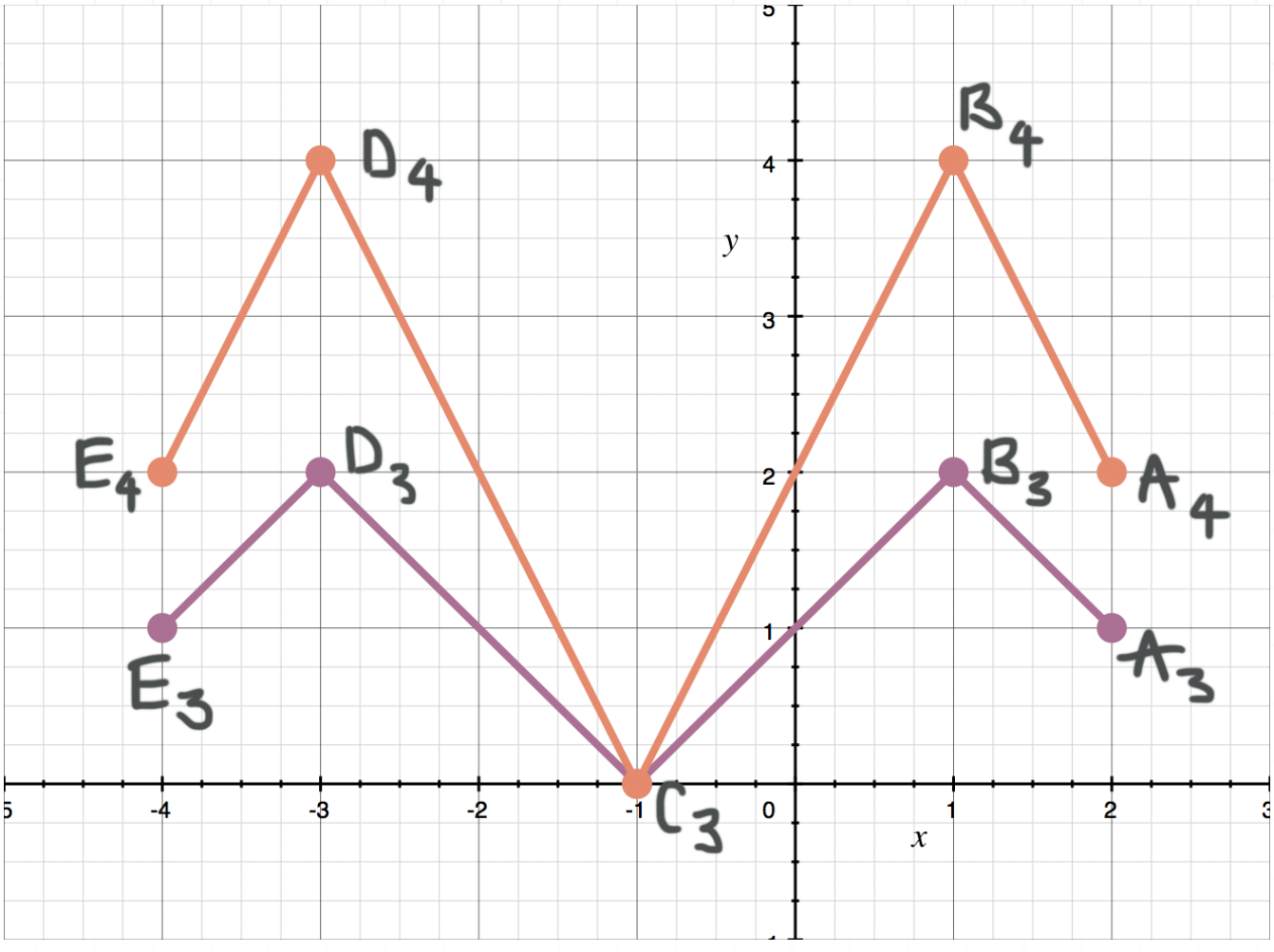
Apply a vertical stretch by 2 by multiplying the y-coordinates by 2.



Apply a horizontal shift to the left by 1 by decreasing the x-coordinates of the points by 1.

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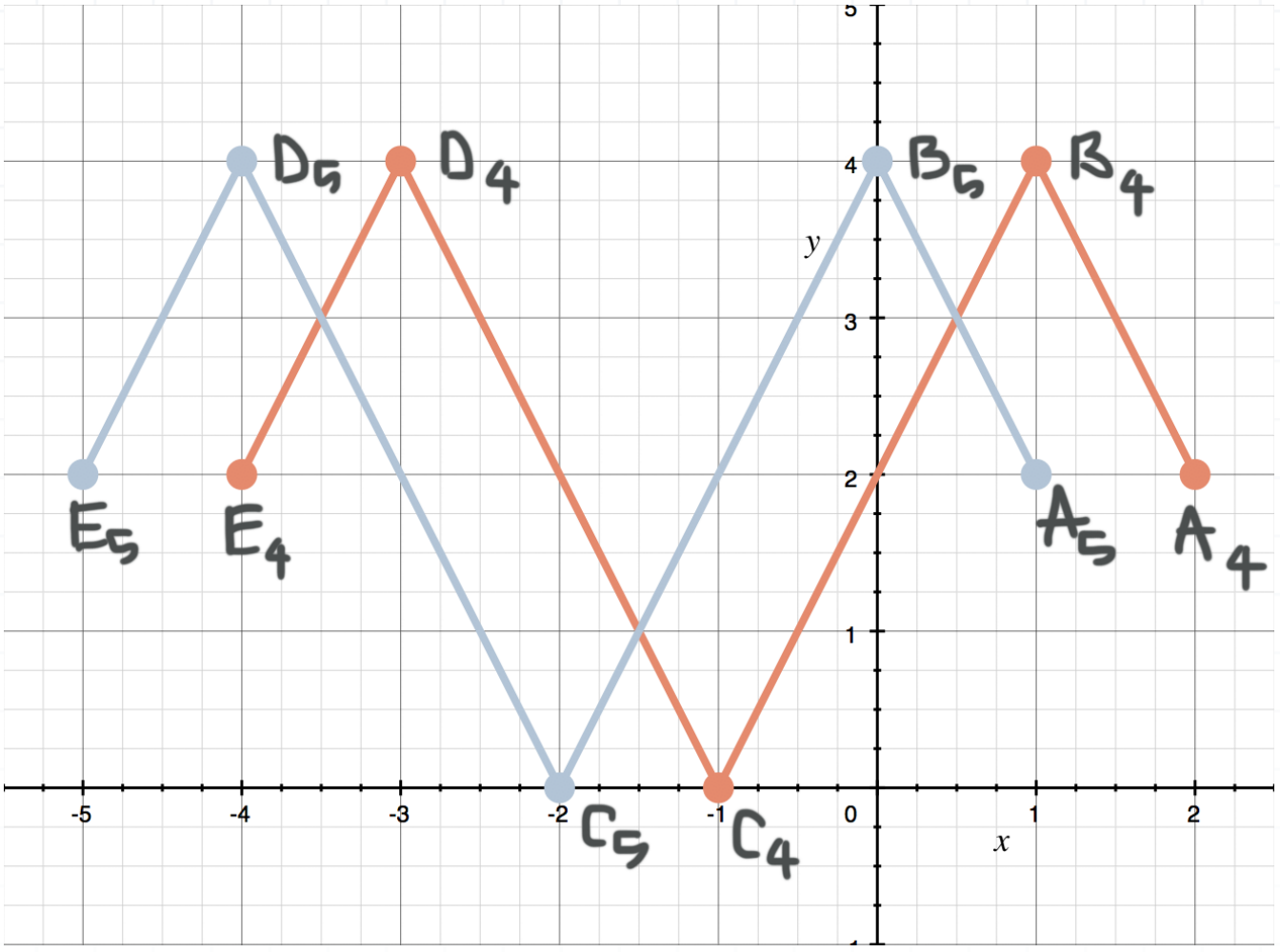
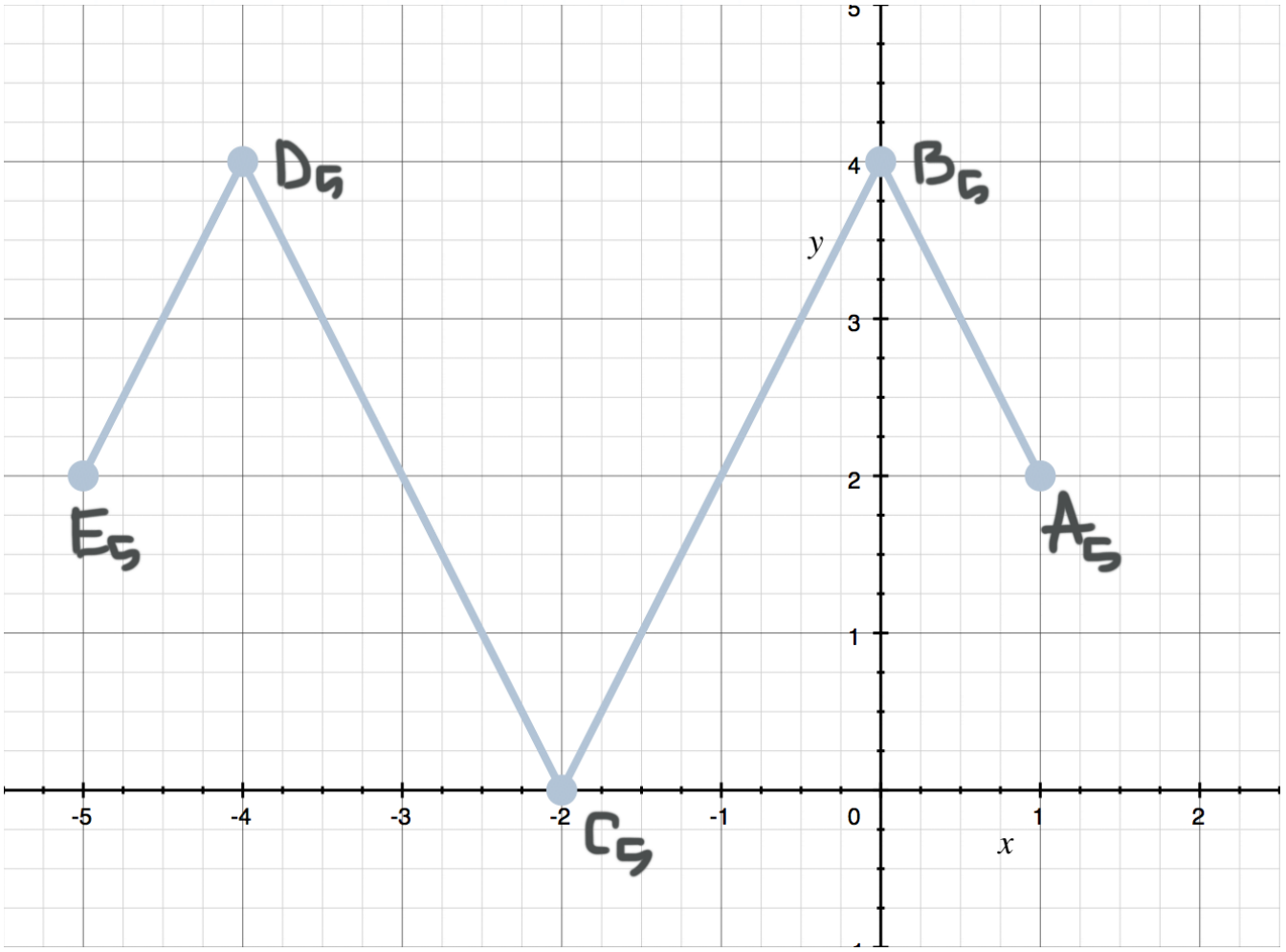


So the graph after the sequence of transformations is



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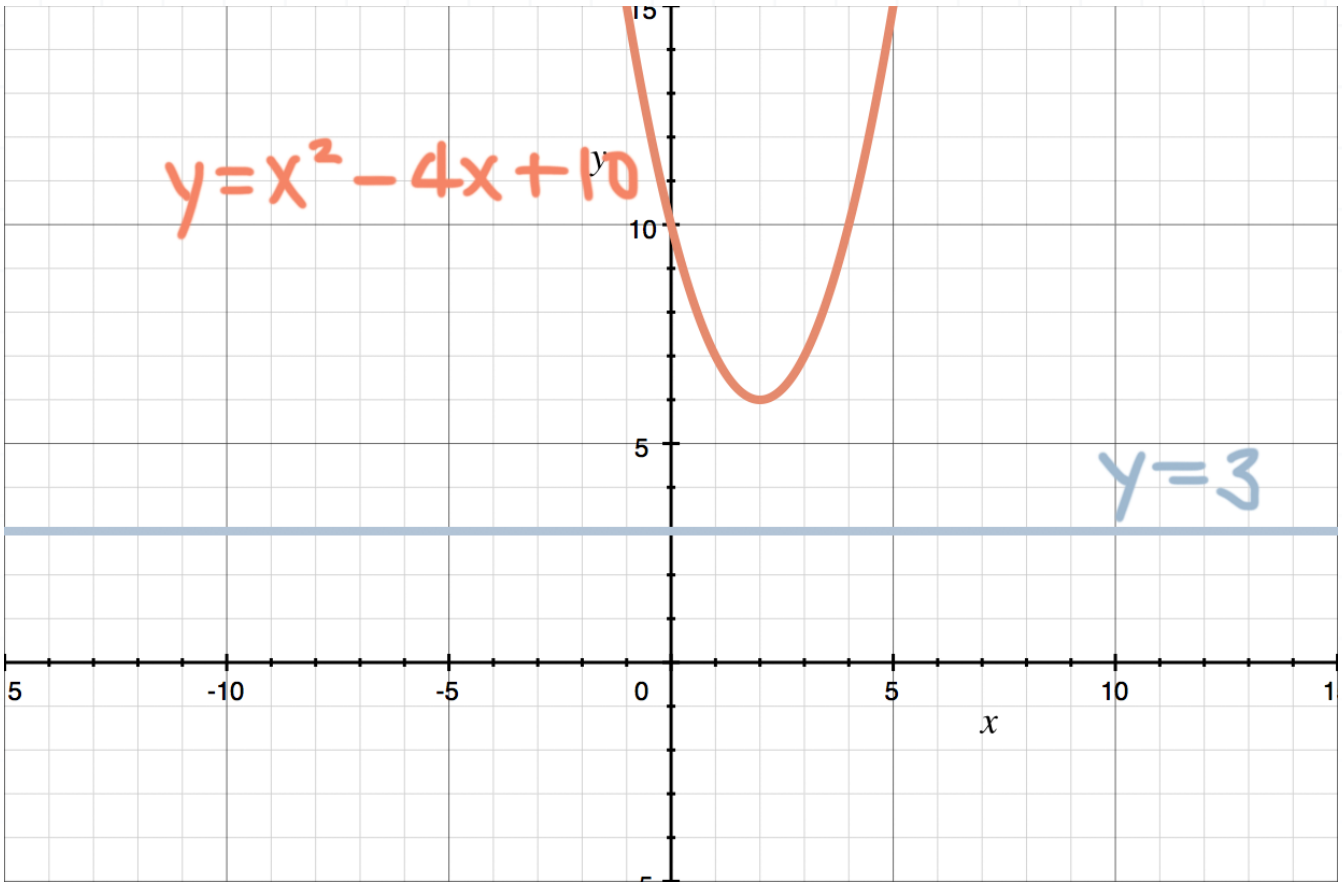
■ 3. Find the equation and sketch the graph of the function obtained by compressing the graph of $f(x) = x^2 - 4x + 10$ with respect to the line $y = 3$ by a factor of 3.



Solution:

In a general case, the compression with respect to the x-axis (which we usually call just "compression"), and the compression with respect to another line, give different results.

For example, consider the vertex of the parabola, $(2,6)$. To compress this point with respect to the x-axis by a factor of 3, we divide the y-coordinate of the point by 3 and get $(2,2)$. At the same time, the compression of $(2,6)$ with respect to $y = 3$ by a factor of 3 gives us $(2,4)$.

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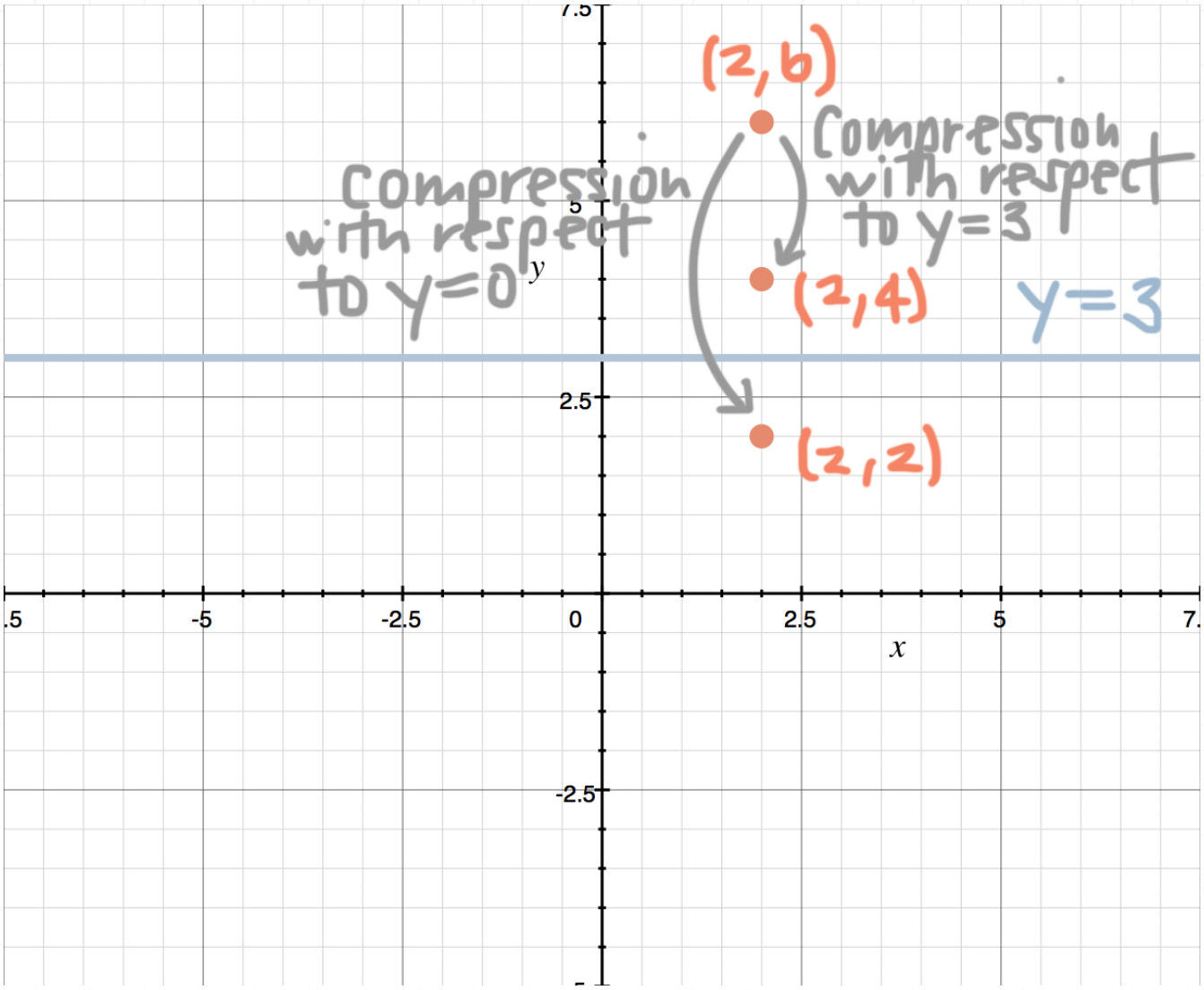

The rules for a compression (stretching) of $f(x)$ with respect to the line $y = a$, with $a > 0$, by a factor of n is

(1) Shift the graph down by a.

(2) Compress (stretch) the graph by a factor of n.

(3) Shift the graph up by a.

For $a < 0$, just swap steps (1) and (3). For $a = 3$, $n = 3$, and $f(x) = x^2 - 4x + 10$, this set of steps gives:

(1) Apply a vertical shift down by 3 by subtracting 3 from the function.

$$f_1(x) = f(x) - 3$$

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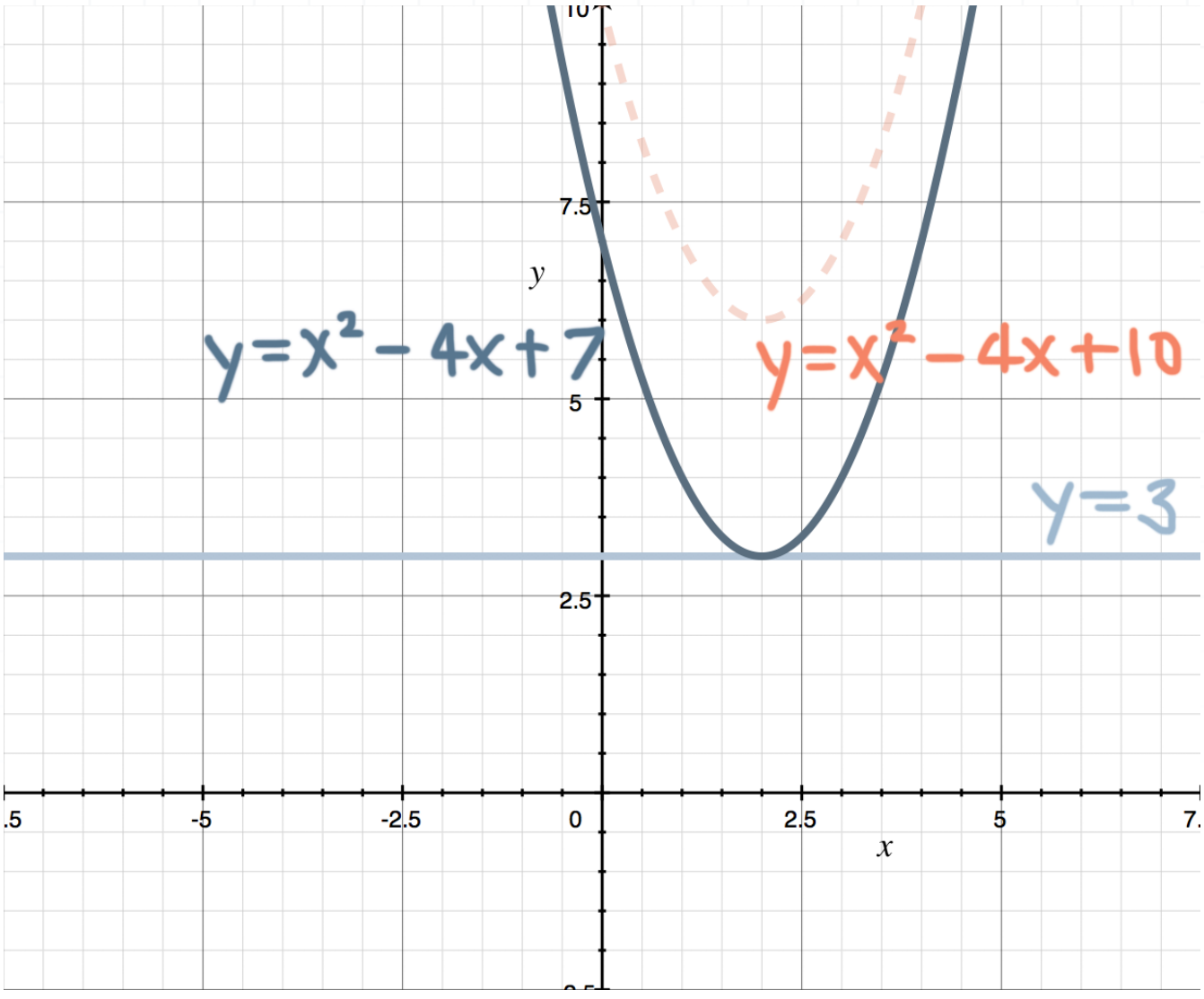
$$f_1(x) = x^2 - 4x + 10 - 3$$

$$f_1(x) = x^2 - 4x + 7$$

A sketch of the transformation is



(2) Apply a compression by 3 by dividing the function by 3.

$$f_2(x) = \frac{f_1(x)}{3}$$

$$f_2(x) = \frac{x^2 - 4x + 7}{3}$$

$$f_2(x) = \frac{1}{3}x^2 - \frac{4}{3}x + \frac{7}{3}$$

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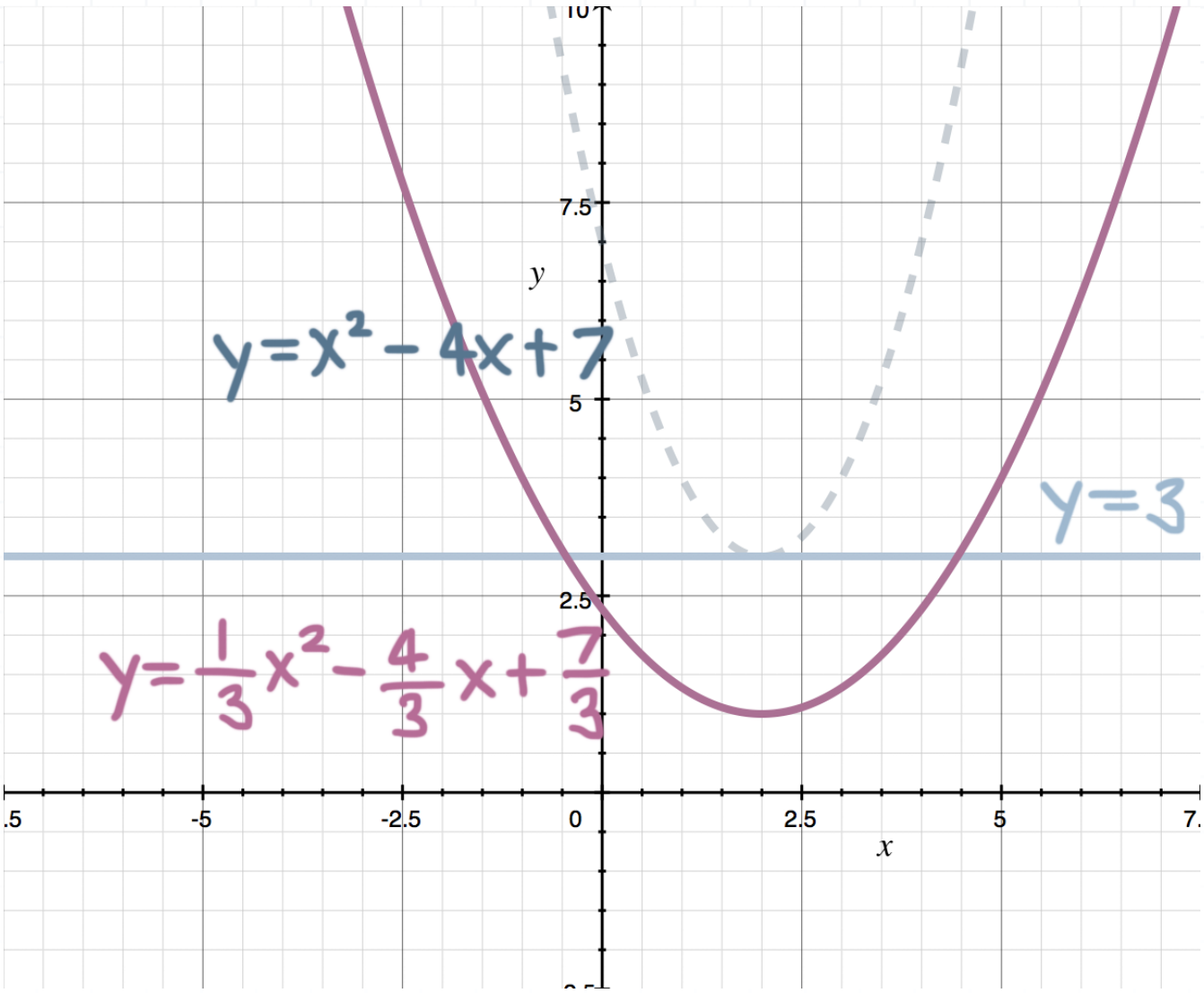
A sketch of the transformation is



(3) Apply a shift up by 3 by adding 3 to the function.

$$f_3(x) = f_2(x) + 3$$

$$f_3(x) = \frac{1}{3}x^2 - \frac{4}{3}x + \frac{7}{3} + 3$$

$$f_3(x) = \frac{1}{3}x^2 - \frac{4}{3}x + \frac{16}{3}$$

So the transformed function is

$$y = \frac{1}{3}x^2 - \frac{4}{3}x + \frac{16}{3}$$

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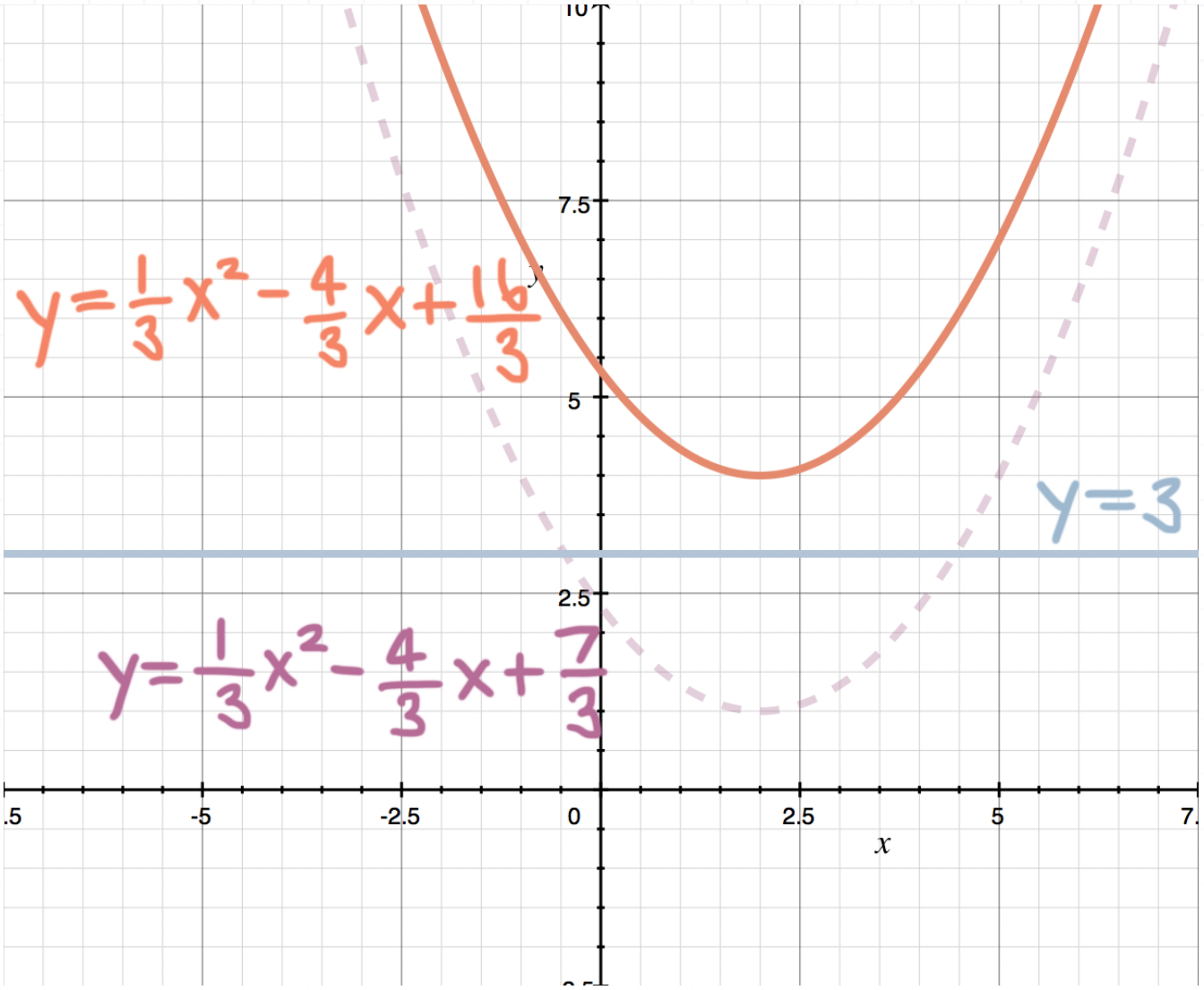
and a sketch of it is



Or in relation to the original function, the sketch is

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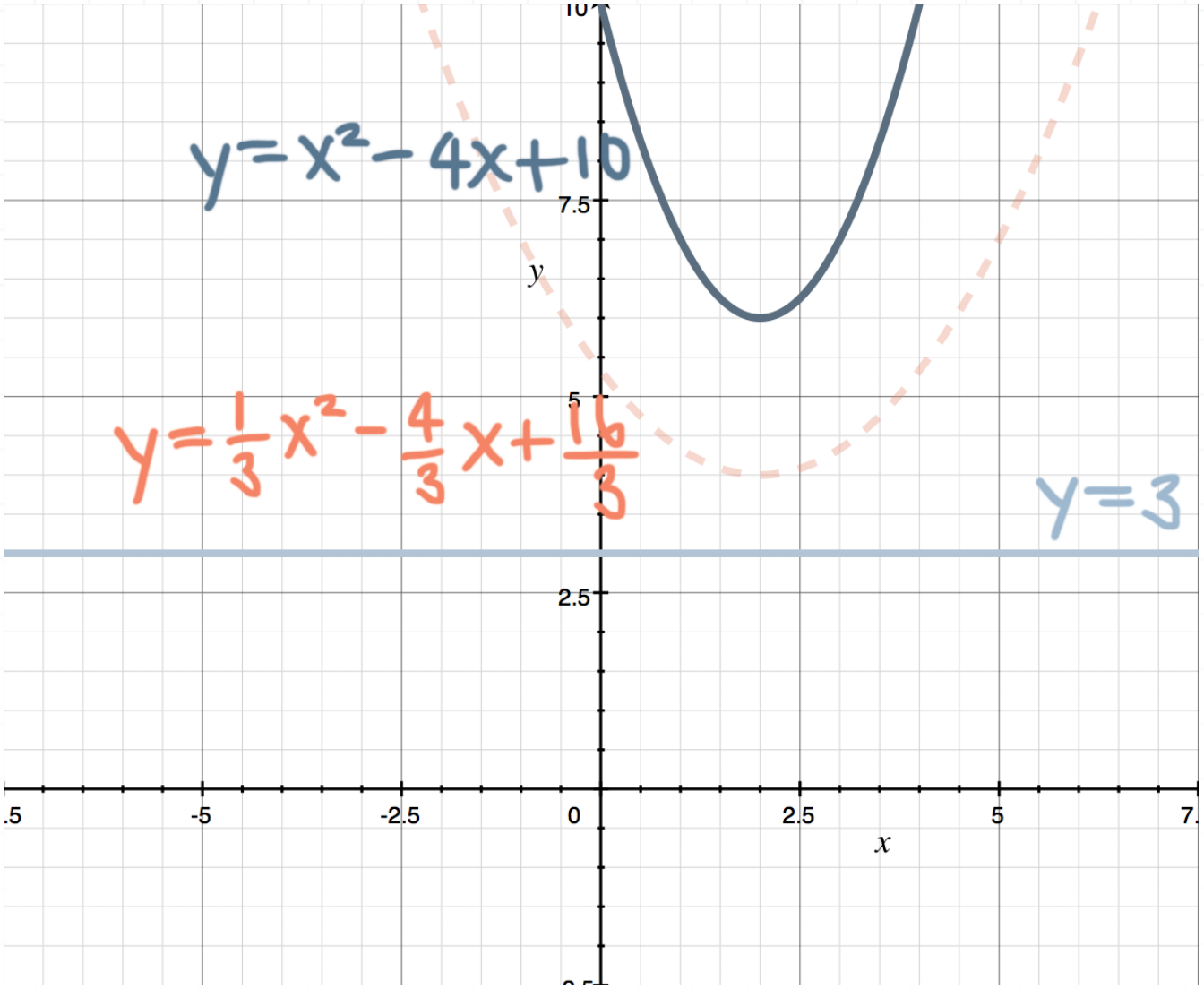

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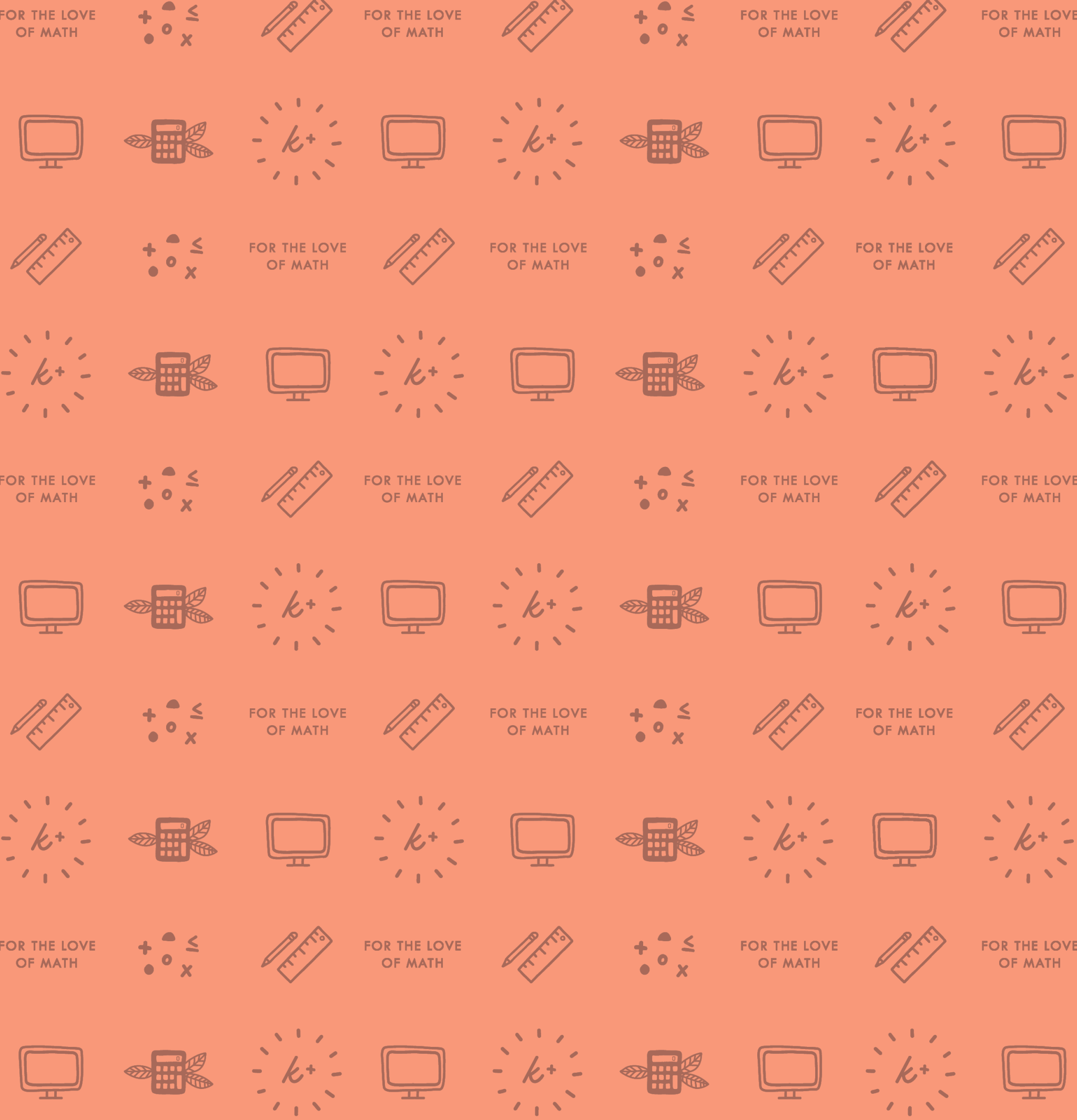
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